3 The link between the stochastic and the deterministic approach

The four stochastic models and the four deterministic models are linked to each other in pairs. First, we prove existence and uniqueness of the solutions to the deterministic systems and show for each pair of stochastic and deterministic models, that the stochastic process converges weakly towards that solution to the deterministic system (Theorems 3.1, 3.3, 3.5 and 3.15). Secondly, we prove in the non-linear cases that the deterministic solutions give good approximations to the stochastic systmes over any finite time interval, if the number of individuals is large (Theorems 3.2 and 3.12). Finally, as already announced in the introduction, in the linear models, the solution to the deterministic model is the expectation of the stochastic model. This has been proven in Theorems 3.4 and 3.21.

3.1 The non-linear models without mortality of humans

We use the notation introduced in chapter 1. The two non-linear models without mortality of humans are models SN and DN. We show how these two models are linked to each other. In Theorems 3.1 and 3.2 we work with $x^{(M,0)}$ with transition rates as in SN and $\xi^{(0)}$ behaving according to DN. The first result shows existence and uniqueness of the solution to the deterministic system DN and that the stochastic process SN converges weakly towards that solution.

Theorem 3.1 [Barbour and Kafetzaki (1993), Theorem 3.2] *Fix $T > 0$. Let $C_1[0,T]$ denote the space of continuous functions f on $[0,T]$ which satisfy $0 \leq f(t) \leq 1$ for all t, and let $C_T^\infty := (C_1[0,T])^\infty$. Suppose that $y \in [0,1]^\infty$ and that $\sum_{j\geq 0} y_j = 1$. Then there is a unique element $\xi^{(0)} \in C_T^\infty$ satisfying the equations DN such that $\xi^{(0)}(0) = y$ and for all $t \in [0,T]$: $\sum_{j\geq 0} \xi_j^{(NL_0)}(t) \leq 1$. Furthermore, if $x^{(M,0)}(0) = y^M$ a.s., where $y^M \to y$ in $[0,1]^\infty$, then $x^{(M,0)}$ converges weakly towards $\xi^{(0)} \in C_T^\infty$.*

Remark: Since T is arbitrarily chosen, it follows that $x^{(M,0)}$ converges weakly towards $\xi^{(0)}$ in $(C_1[0,\infty))^\infty$, where $C_1[0,\infty)$ has the projective limit topology. It is also in order to allow the initial values $x^{(M,0)}(0)$ to be random, provided that the sequence of random elements $x^{(M,0)}(0)$ of $[0,1]^\infty$ converges weakly to y.

As announced above the behaviour of the stochastic process *in a finite time interval* is much the same as the behaviour of the solution of the corresponding differential equations if the number of individuals is large. This is the content of

Theorem 3.2 [Barbour and Kafetzaki (1993), Theorem 3.5] *Suppose that* $y \in [0,1]^\infty$ *is such that* $\sum_{j\geq 0} y_j = 1$ *and* $s_\alpha := \sum_{j\geq 1} j^\alpha y_j < \infty$ *for some* $\alpha > 0$. *Then, if* $\xi^{(0)}$ *is the solution of DN with* $\xi^{(0)}(0) = y$, $\sum_{j\geq 0} \xi_j^{(0)}(t) = 1$ *for all* t. *If also* $x^{(M,0)}(0) = y^M \to y$ *in such a way that* $\lim_{M\to\infty} \sum_{j\geq 0} j^\alpha y_j^M = s_\alpha$, *then*

$$\lim_{M\to\infty} \mathbb{P}[\sup_{0\leq t\leq T} \sum_{j\geq 0} |x_j^{(M,0)}(t) - \xi_j^{(0)}(t)| > \epsilon] = 0$$

for any $T, \epsilon > 0$.

3.2 The linear models without mortality of humans

We again use the notation introduced in chapter 1. The two linear models without mortality of humans are models SL and DL. So we present how these two models are linked to each other. In Theorems 3.3 and 3.4 we work with $X^{(0)}$ with transition rates as in SL and $\Xi^{(0)}$ behaving according to DL. The first result shows that the (normalised) stochastic process SL converges weakly towards the solution to the deterministic system DL. We need sequences of processes behaving according to SL. We denote them with $(X^{(K,0)}, K \geq 1)$.

Theorem 3.3 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.1] *Let* $(X^{(K,0)}, K \geq 1)$ *be a sequence of Markov branching processes as specified in SL. We presume the initial state* $X^{(K,0)}(0)$ *is such that* $\sum_{j\geq 1} X_j^{(K,0)}(0) < \infty$ *and* $K^{-1}X^{(K,0)}(0) \to y^{(0)}$, *where* $0 < \sum_{j\geq 1} jy_j^{(0)} < \infty$. *Then* $K^{-1}X^{(K,0)}$ *converges weakly in* $D^\infty[0,T]$ *for each* $T > 0$ *to a nonrandom, nonnegative process* $\Xi^{(0)}$, *which evolves according to the differential equations DL with initial state* $\Xi^{(0)}(0) = y^{(0)}$, *and satisfies conditions C.*

Remarks 1. That such a solution $\Xi^{(0)}$ to DL with given initial values exists and is unique is proven in Theorem 4.9. We do not use Theorem 3.3 to prove Theorem 4.9.

2. We can loosen the condition $0 < \sum_{j\geq 1} jy_j^{(0)} < \infty$ to $0 < \sum_{j\geq 1} y_j^{(0)} < \infty$. Then conditions C are not necessarily satisfied and all we can guarantee (of conditions C) is that $\sup_{0\leq s\leq t} \sum_{j\geq 1} \Xi_j^{(0)}(s) < \infty$ for all $t \geq 0$ and that there exists a $j \geq 1$ such that $\Xi_j^{(0)}(0) > 0$.

3. As $\Xi^{(0)}$ in Theorem 3.3 is the weak limit of nonnegative stochastic processes, $\Xi^{(0)}$ is nonnegative too.

In the linear cases the solution to the deterministic model is the expectation of the stochastic model. The first result with $\kappa = 0$ is

Theorem 3.4 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.2] *Let $X^{(0)}$ be a Markov process with rates given in SL and with initial state $X^{(0)}(0)$ satisfying $\sum_{j\geq 1} X_j^{(0)}(0) =: M < \infty$, and set $\Xi^{(0)}(0) := M^{-1}X^{(0)}(0)$. Then y defined by*

$$y_j(t) := M^{-1}\mathbb{E}\{X_j^{(0)}(t)|X^{(0)}(0) = M\Xi^{(0)}(0)\}$$

satisfies the differential equations DL with $y(0) = \Xi^{(0)}(0)$, as well as conditions C.

3.3 The non-linear models with mortality of humans

Again we use the notation introduced in chapter 1. The two non-linear models with mortality of humans are models SNM and DNM. We show how these models are linked to each other. In Theorems 3.5 and 3.12 we work with $x^{(M)}$ with transition rates as in SNM and ξ behaving according to DNM. The first result shows existence and uniqueness of the solution to the deterministic system DNM and that the stochastic process SNM converges weakly towards that solution.

Theorem 3.5 *Fix $T > 0$. Let $C_1[0,T]$ denote the space of continuous functions f on $[0,T]$ which satisfy $0 \leq f(t) \leq 1$ for all t, and let $C_T^\infty := (C_1[0,T])^\infty$. Suppose that $y \in [0,1]^\infty$ and that $\sum_{j\geq 0} y_j = 1$. Then there is a unique element $\xi \in C_T^\infty$ satisfying the equations DNM such that $\xi(0) = y$ and for all $t \in [0,T]:\ \sum_{j\geq 0}\xi_j(t) \leq 1$. Furthermore, if $x^{(M)}(0) = y^M$ a.s., where $y^M \to y$ in $[0,1]^\infty$, then $x^{(M)}$ converges weakly towards $\xi \in C_T^\infty$.*

Remark to Theorem 3.5: Since T is arbitrarily chosen, it follows that $x^{(M)}$ converges weakly towards ξ in $(C_1[0,\infty))^\infty$, where $C_1[0,\infty)$ has the projective limit topology. It is also in order to allow the initial values $x^{(M)}(0)$ to be random, provided that the sequence of random elements $x^{(M)}(0)$ of $[0,1]^\infty$ converges weakly to y.

The proof of Theorem 3.5 consists of several parts. We first prove Lemma 3.6, because it is interesting on its own and already proves one part of Theorem 3.5; we also need it to prove the rest of Theorem 3.5. After proving Lemma 3.6, some notation is introduced and the strategy of the proof is sketched.

Lemma 3.6 *Given the initial values $\xi(0)$ of a possible solution of DNM, there is at most one solution of DNM which satisfies $\sum_{l\geq 0}\xi_l(t) \leq 1$ for all $t \geq 0$.*

Proof of Lemma 3.6 System DNM can be rewritten in integral form as equations (3.1):

$$\begin{aligned} \xi_j(t) =& \xi_j(0) + \int_0^t \Big\{ (j+1)\xi_{j+1}(u)\mu - j\xi_j(u)\mu \\ &+ \lambda\xi_0(u)\sum_{l\geq 1}\xi_l(u)p_{lj} - \kappa\xi_j(u)\Big\} du,\ j \geq 1; \\ \xi_0(t) =& \xi_0(0) + \int_0^t \Big\{ \xi_1(u)\mu \\ &- \lambda\xi_0(u)(1 - \sum_{l\geq 0}\xi_l(u)p_{l0}) + \kappa(1-\xi_0(u))\Big\} du,\ (j=0). \end{aligned} \tag{3.1}$$

We prove Lemma 3.6 by showing that each solution of (3.1) (and therefore of DNM) has the same Laplace transform, and then applying the uniqueness theorem (see Feller (1966) for example). To obtain the Laplace transform, we multiply the j equation in (3.1) by e^{-js}, for any fixed $s > 0$, and add over $j \geq 0$, obtaining

$$\begin{aligned} \phi(s,t) =& \phi(s,0) + \int_0^t \Big\{ \mu(1-e^s)\frac{\partial\phi(s,u)}{\partial s} \\ &+ \lambda\phi(\infty,u)[\phi(-\log\psi(s),u) - 1] + \kappa - \kappa\phi(s,u)\Big\} du; \end{aligned}$$

where $\phi(s,t) := \sum_{j\geq 0} e^{-js}\xi_j(t)$ and $\psi(s) := \sum_{j\geq 0} e^{-js}p_{1j}$; $\phi(\infty,t)$ is just another way of writing $\xi_0(t)$. Differentiating with respect to t leads to the partial differential equation

$$\frac{\partial\phi(s,t)}{\partial t} = \mu(1-e^s)\frac{\partial\phi(s,t)}{\partial s} + \lambda\phi(\infty,t)[\phi(-\log\psi(s),t)-1] + \kappa - \kappa\phi(s,t). \tag{3.2}$$

Equation (3.2) can be integrated in $s > 0, t \geq 0$, using the concept of characteristics (see Courant and Hilbert (1968), chapter II, §1 for example): Equation (3.2) is a quasi-linear differential equation of first order with two independent variables. It can be rewritten in the form

$$a\phi_t + b\phi_s = c,$$

where the notation is such that ϕ_i means the derivative with respect to i where $i \in \{s,t\}$ and $a := 1$, $b := \mu(e^s - 1)$ and $c := \lambda\phi(\infty,t)[\phi(-\log\psi(s),t) - 1] + \kappa - \kappa\phi(s,t)$. The characteristic differential equations are the following three:

$$\begin{aligned} \frac{dt}{dx} &= a \\ \frac{ds}{dx} &= b \\ \frac{d\phi}{dx} &= c. \end{aligned}$$

As $a = 1$ we have $dt = dx$ and therefore

$$\frac{ds}{dt} = \mu(e^s - 1)$$
$$\frac{d\phi}{dt} = \lambda\phi(\infty, t)[\phi(-\log\psi(s), t) - 1] + \kappa - \kappa\phi(s, t).$$

The solution $S(t)$ to the first equation is $S(t) = -\log(1 - e^{\mu t + K})$ for some constant K. For initial values t_0, s_0 we have $s_0 = S(t_0)$ and so we can calculate that $K = \log(1 - e^{-s_0}) - t_0\mu$. The general solution to the first equation is therefore

$$S_{s_0,t_0}(t) = -\log\{1 - (1 - e^{-s_0})e^{-\mu(t_0 - t)}\}.$$

So we have

$$\phi(s,t) = \phi(S_{s,t}(v), v) + \int_v^t \lambda\phi(\infty, u)[\phi(-\log\psi(S_{s,t}(u)), u) - 1] + \kappa - \kappa\phi(S_{s,t}(u), u)\, du; \tag{3.3}$$

for any v, and in particular for $v = 0$.

Now if ξ_1 and ξ_2 are two different solutions of (3.1), they give rise to functions ϕ_1 and ϕ_2 satisfying (3.3), and such that $0 \leq \phi_i \leq 1$, $i = 1, 2$. Suppose that, for any $v \geq 0$, $\phi_1(s, v) = \phi_2(s, v)$ for all s (as is certainly the case for $v = 0$). Let

$$d_{v,w}(\phi_1, \phi_2) := \sup_{v \leq t \leq w} \sup_{s > 0} |\phi_1(s,t) - \phi_2(s,t)| \leq 1.$$

Then, from (3.3), for $t \in [v, w]$,

$$|\phi_1(s,t) - \phi_2(s,t)| = \Big| \int_v^t \Big\{ \lambda\phi_1(\infty, u)[\phi_1(-\log\psi(S_{s,t}(u)), u) - 1]$$
$$+\kappa - \kappa\phi_1(S_{s,t}(u), u) - \lambda\phi_2(\infty, u)[\phi_2(-\log\psi(S_{s,t}(u)), u) - 1]$$
$$-\kappa + \kappa\phi_2(S_{s,t}(u), u) \Big\} du \Big| \leq (\kappa + 2\lambda)(w - v)d_{v,w}.$$

But then we have

$$d_{v,w} \leq (\kappa + 2\lambda)(w - v)d_{v,w}.$$

But this in turn implies that $d_{v,w} = 0$ if $w < v + (\kappa + 2\lambda)^{-1}$. Iterating this procedure, starting with $v = 0$ and continuing in steps of $(2(\kappa + 2\lambda))^{-1}$ shows that $\phi_1(s,t) = \phi_2(s,t)$, for all $s > 0, t \geq 0$ which completes the proof of Lemma 3.6.

□

It is convenient to define the following functions and random variables to use in what follows:

$$
\begin{aligned}
a_j(x) &:= (j+1)\mu x_{j+1} - j\mu x_j + \lambda x_0 \sum_{l\geq 1} x_l p_{lj} - \kappa x_j;\ j \geq 1,\\
a_0(x) &:= \mu x_1 - \lambda x_0 (1 - \sum_{l\geq 0} x_l p_{l0}) + \kappa(1 - x_0),\\
b_j(x) &:= (j+1)\mu x_{j+1} + j\mu x_j + \lambda x_0 \sum_{l\geq 1} x_l p_{lj} + \kappa x_j;\ j \geq 1,\\
b_0(x) &:= \mu x_1 + \lambda x_0 (1 - \sum_{l\geq 0} x_l p_{l0}) + \kappa(1 - x_0),\\
a_j^* &:= \sup_x |a_j(x)| \leq (j+1)\mu + \lambda + \kappa < \infty;\ j \geq 1, \qquad (3.4)\\
b_j^* &:= \sup_x |b_j(x)| \leq 2(j+1)\mu + \lambda + \kappa < \infty;\ j \geq 1,\\
U_j(x(t)) &:= x_j(t) - x_j(0) - \int_0^t a_j(x(u))du;\ j \geq 0,\\
U_j^M &:= U_j(x^{(M)}),\ j \geq 0,\\
U_j^* &:= U_j(x^*),\ j \geq 0,\\
V_j^M(t) &:= U_j^M(t)^2 - \frac{1}{M}\int_0^t b_j(x^{(M)}(u))du;\ j \geq 0,
\end{aligned}
$$

where x^* is to be defined later. Further, let $\mathcal{H}_t^M$ denote $\sigma\{x^{(M)}(s), 0 \leq s \leq t\}$.

We need the following lemma to prove Theorem 3.5.

Lemma 3.7 *$U_j^M(t)$ and $V_j^M(t)$ are $\mathcal{H}_t^M$-martingales.*

Proof of Lemma 3.7 This is going to be proved in the Appendix as Corollary A9.

□

Strategy of the proof of Theorem 3.5

The proof consists of two parts: In the first part (Lemmas 3.8 and 3.9) we show that we do indeed have weak convergence. In Lemma 3.8, we show that the sequence $\{x_j^{(M)}\}, M \geq 1$, is tight for each j. Using that, we can prove Lemma 3.9. It shows, that for each subsequence $\bar{N} \subset \mathbb{N}$ of the indexes M there exists a subsubsequence $N \subset \bar{N}$ and a random function $x^* = x^*(N)$ such that $x^{(M)}$ converges weakly towards x^* for $M \in N,\ M \to \infty$. From the point of view of the corresponding measures, Lemma 3.9 just says that the sequence of probability measures is relatively compact. We denote the limit-function by $x^*(N)$ because up until now it is still dependent of N.

In the second part (Lemmas 3.10 and 3.11) we prove that for each choice of N the limit x^* is the unique solution ξ of DNM. In particular, we then have proved that a solution of DNM exists. That ξ is unique we proved in Lemma 3.6. Using Theorem 2.3 of Billingsley (1968) we have therefore shown weak convergence.
In Lemma 3.10 we show that U_j^M converges weakly towards U_j^*; and in Lemma 3.11 and the remarks to Lemma 3.11 we show, that indeed x^* is a solution to DNM.

The methods used are mixtures of the proofs of the analogous theorems of Barbour and Kafetzaki (1993), Kafetzaki (1993) and Barbour, Heesterbeek and Luchsinger (1996).

Proof of Theorem 3.5 Take y as in the statement of the theorem, and choose a sequence y^M of deterministic initial conditions for $x^{(M)}$ such that $y^M \to y$ in $[0,1]^\infty$. Fix any j, and consider the uniform modulus of continuity

$$w(x_j^{(M)};\delta) := \sup_{0\le s\le t\le T;t-s<\delta} |x_j^{(M)}(t) - x_j^{(M)}(s)|$$

of the random elements $x_j^{(M)}$ of the space $D[0,T]$, given the Skorohod topology. First we need to prove the following

Lemma 3.8 *For each j, the sequence $\{x_j^{(M)}\}_{M\ge1}$ is tight in $D[0,T]$ and any weak limit belongs to $C[0,T]$.*

Proof of Lemma 3.8: Let us denote by P_M the probability-measure of $x_j^{(M)}$. By definition a sequence of random elements $x_j^{(M)}, M \ge 1$, is tight if the sequence of the corresponding probability-measures $P_M, M \ge 1$, is tight. We are going to apply Billingsley (1968), Theorem 15.5, for which we need to check, that the following two conditions are satisfied:

1) For all $\eta > 0 \; \exists \; a$, such that

$$P_M\{x : x(0) > a\} \le \eta,$$

for all $M \ge 1$, and

2) For all $(\epsilon, \eta) > (0,0)$, $\exists \; \delta : 0 < \delta < 1$, and a natural number M_0, such that

$$P_M\{x : w(x;\delta) \ge \epsilon\} \le \eta,$$

for all $M \geq M_0$.

In Theorem 15.5 of Billingsley (1968) it is shown that if this is true, then we additionally have $P(C) = 1$ for P any weak limit of a subsequence $\{P_{M'}\}$, where C denotes as usual the space of the continuous functions.

So we only have to prove the two conditions above.

Condition 1) is satisfied because by construction of $x^{(M)}$, we have $x_j^{(M)}(0) \leq 1$ $P_M - a.s.$ for all M, j.

All that remains is condition 2). From the definition of U_j^M, it follows that

$$\begin{aligned} |x_j^{(M)}(t) - x_j^{(M)}(s)| &\leq |U_j^M(t) - U_j^M(s)| + \int_s^t |a_j(x^{(M)}(u))|du \\ &\leq |U_j^M(t) - U_j^M(s)| + (t-s)a_j^*. \end{aligned} \tag{3.5}$$

Now by the Doob-Kolmogorov inequality for martingales (see Stroock (1993), page 355) and because U_j^M and V_j^M are martingales by Lemma 3.7, it follows that for s arbitrary but fixed,

$$\begin{aligned} \mathbb{P}[\sup_{s<t\leq s+\delta} |U_j^M(t) - U_j^M(s)| \geq \frac{\epsilon}{6}] &\leq \frac{36}{\epsilon^2}\mathbb{E}[(U_j^M(s+\delta) - U_j^M(s))^2] \\ &= \frac{36}{\epsilon^2}\Big\{\mathbb{E}[U_j^M(s+\delta)^2] - \mathbb{E}[U_j^M(s)^2]\Big\} \\ &= \frac{36}{\epsilon^2}\Big\{\mathbb{E}\big[V_j^M(s+\delta) + \frac{1}{M}\int_0^{s+\delta} b_j(x^{(M)}(u))du\big] \\ &\quad -\mathbb{E}\big[V_j^M(s) + \frac{1}{M}\int_0^s b_j(x^{(M)}(u))du\big]\Big\} \\ &= \frac{36}{\epsilon^2 M}\mathbb{E}[\int_s^{s+\delta} b_j(x^{(M)}(u))du] \leq \frac{36\delta b_j^*}{\epsilon^2 M}. \end{aligned} \tag{3.6}$$

Hence, given $\epsilon, \eta > 0$ pick δ so small that $\delta a_j^* < \epsilon/6$ with $\delta = T/n$ for some integer n. We then choose M_0 so large that, for all $M > M_0$, $36b_j^*/(\epsilon^2 M) < \eta/T$, so that, from (3.5) and (3.6),

$$\begin{aligned} &\mathbb{P}[\sup_{s<t\leq s+\delta} |x_j^{(M)}(t) - x_j^{(M)}(s)| \geq \frac{\epsilon}{3}] \\ &\qquad \leq \mathbb{P}[\sup_{s<t\leq s+\delta} \{|U_j^M(t) - U_j^M(s)| + (t-s)a_j^*\} \geq \frac{\epsilon}{3}] \\ &\qquad \leq \mathbb{P}[\sup_{s<t\leq s+\delta} |U_j^M(t) - U_j^M(s)| \geq \frac{\epsilon}{6}] + \mathbb{P}[\delta a_j^* > \frac{\epsilon}{6}]. \end{aligned}$$

The second term above is 0 because we have chosen δ so small that $\delta a_j^* < \epsilon/6$. We therefore have:

$$\mathbb{P}[\sup_{s<t\leq s+\delta} |x_j^{(M)}(t) - x_j^{(M)}(s)| \geq \frac{\epsilon}{3}] \leq \frac{36\delta b_j^*}{\epsilon^2 M} < \frac{\eta\delta}{T}, \tag{3.7}$$

for $M > M_0$ and for any *fixed* $s \in [0, T]$. Using (3.7) we derive that $\mathbb{P}[A_s^M] \leq \eta\delta/T$ for all $s \in [0, T]$, where $A_s^M := \{x_j^{(M)} : \sup_{s<t\leq s+\delta} |x_j^{(M)}(t) - x_j^{(M)}(s)| \geq \epsilon/3\}$. Therefore we have

$$\mathbb{P}[\{x_j^{(M)} : w(x_j^{(M)}; \delta) \geq \epsilon\}] \leq \mathbb{P}[\cup_{i=0}^{n-1} A_{i\delta}^M] \leq \sum_{i=0}^{n-1} \mathbb{P}[A_{i\delta}^M] \leq \frac{n\eta\delta}{T} = \eta.$$

The first inequality is justified because of Billingsley (1968), equation (8.6), page 56. Therefore condition 2) is satisfied too which ends the proof of Lemma 3.8.

□

Lemma 3.9 *Given any infinite subsequence $\bar{N} \subset \mathbb{N}$, there exists a subsequence $N \subset \bar{N}$ such that $x^{(M)}$ converges weakly in $(D[0, T])^\infty$ along N. We denote the limit by $x^* := x^*(N)$.*

Remark to Lemma 3.9 From the viewpoint of probability measures P_M - instead of random elements $x^{(M)}$ - the content of Lemma 3.9 is the statement that the family of probability measures $\{P_M\}_{M\geq 1}$ is relatively compact.

Proof of Lemma 3.9: It is enough by Prohorov's Theorem (Billingsley (1968, Theorem 6.1)) to show that the sequence $x^{(M)}$ is tight in $(D[0, T])^\infty$. Given $\epsilon > 0$, let $K_j,\ j \geq 0$, be a compact set in $D[0, T]$ such that $\mathbb{P}[x_j^{(M)} \in K_j] > 1 - 2^{-(j+1)}\epsilon$. Such a K_j exists because of Lemma 3.8. Then $K := \prod_{j\geq 0} K_j$ is compact in $(D[0, T])^\infty$, and $\mathbb{P}[x^{(M)} \in K] > 1 - \epsilon$.

□

Lemma 3.10 U_j^M *converges weakly towards* U_j^* *in* $D[0, T],\ M \in N,\ j \geq 0$.

Proof of Lemma 3.10: We prove Lemma 3.10 by showing that U_j (see (3.4)) is continuous at x^*. Let $(z_n, n \geq 0)$ be a sequence of elements of $(D[0, T])^\infty$, such that $\lim_{n\to\infty} z_n = z \in C_T^\infty$ and $0 \leq z_{nj}(t) \leq 1$ for all (n, j, t). Then, since convergence in $D[0, T]$ to an element of $C[0, T]$ implies uniform convergence we have: $\lim_{n\to\infty} \sup_{0\leq t\leq T} |z_{nl}(t) - z_l(t)| = 0$; for all $l \geq 0$. So all we have to concentrate on to show continuity of U_j is the part

$z_{n0}(t)\sum_{l\geq 0} z_{nl}(t)p_{lj}$ of the integral. But here we can proceed in the following way:

$$\sup_{0\leq t\leq T} |z_{n0}(t)\sum_{l\geq 0} z_{nl}(t)p_{lj} - z_0(t)\sum_{l\geq 0} z_l(t)p_{lj}|$$

$$\leq \{\sup_{0\leq t\leq T} |z_{n0}(t)-z_0(t)|\sum_{l\geq 0} p_{lj}\} + \{\sum_{l\geq 0}\sup_{0\leq t\leq T} |z_{nl}(t)-z_l(t)|p_{lj}\}. \quad (3.8)$$

But by Lemma 2.1 we have $\sum_{l\geq 0} p_{lj} < \infty$, and using the dominated convergence theorem, we see that (3.8) converges to 0 as $n\to\infty$. This ends the proof of Lemma 3.10.

□

Lemma 3.11 *For all $j\geq 0,\ \epsilon>0$ we have*

$$\mathbb{P}[\sup_{0\leq t\leq T} |U_j^M(t)| > \epsilon] \to 0,$$

$M\to\infty,\ M\in N.$

Proof of Lemma 3.11: By the Doob-Kolmogorov inequality for martingales applied to U_j^M and from the martingale property of V_j^M, it follows that

$$\mathbb{P}[\sup_{0\leq t\leq T} |U_j^M(t)| > \epsilon] \leq \frac{1}{\epsilon^2}\mathbb{E}[U_j^M(T)^2] = \frac{1}{\epsilon^2 M}\mathbb{E}[\int_0^T b_j(x^{(M)}(u))du]$$

$$\leq \frac{Tb_j^*}{\epsilon^2 M} \to 0,\ M\to\infty,\ M\in N.$$

This ends the proof of Lemma 3.11.

□

Final remarks to the proof of Theorem 3.5 As a consequence of Lemmas 3.10 and 3.11 we therefore have

$$\mathbb{P}[x_j^*(t) = x_j^*(0) + \int_0^t a_j(x^*(u))du,\ \text{for all}\ t:\ 0\leq t\leq T] = 1$$

for each weak limit x^*. So x^* satisfies DNM, that is there exists at least one solution of DNM. But Lemma 3.6 shows that there is at most one solution that satisfies the conditions asked for in Theorem 3.5. Also, because $x^{(M)}$ converges weakly towards x^* we have $\sum_{j\geq 0} x_j^*(t) \leq 1$; for all t a.e.. So, using Theorem 2.3 of Billingsley (1968) we have proven Theorem 3.5. As the solution x^* to DNM is the weak limit of nonnegative stochastic processes, that solution x^* to DNM stays nonnegative too (as required).

□

Here too, the behaviour of the stochastic process *in a finite time interval* is much the same as the behaviour of the solution of the corresponding differential equations if the number of individuals is large. This is the content of

Theorem 3.12 *Suppose that* $y \in [0,1]^\infty$ *is such that* $\sum_{j\geq 0} y_j = 1$ *and* $s_\alpha := \sum_{j\geq 1} j^\alpha y_j < \infty$ *for some* $\alpha > 0$. *Then, if* ξ *is the solution of DNM with* $\xi(0) = y$, $\sum_{j\geq 0} \xi_j(t) = 1$ *for all* t. *If also* $x^{(M)}(0) = y^M \to y$ *in such a way that* $\lim_{M\to\infty} \sum_{j\geq 0} j^\alpha y_j^M = s_\alpha$, *then*

$$\lim_{M\to\infty} \mathbb{P}[\sup_{0\leq t\leq T} \sum_{j\geq 0} |x_j^{(M)}(t) - \xi_j(t)| > \epsilon] = 0$$

for any $T, \epsilon > 0$.

Remark This result is somewhat stronger than Theorem 3.5 but we have to impose slight restrictions on the initial value y of ξ.

Preparations for the proof of Theorem 3.12 The following lemma has been proven in Barbour and Kafetzaki (1993) as Equation (3.20). We only use the first case ($\alpha \leq 1$) in this thesis, but the results are interesting themselves.

Lemma 3.13 [Barbour and Kafetzaki (1993), Equation (3.20)] *Let* α *be greater than 0. Then the following inequalities hold:*

$$\sum_{j\geq 1} j^\alpha p_{ij} \leq \begin{cases} (i\theta)^\alpha, & \alpha \leq 1; \\ (i\theta)^\alpha\{1 + \sigma^2/i\theta^2\}, & 1 \leq \alpha \leq 2; \\ (i\theta)^\alpha\{1 + O(i^{-1}[\sigma^2/\theta^2 + \gamma_\alpha/\theta^\alpha])\}, & \alpha > 2, \end{cases}$$

where $\gamma_\alpha := \sum_{j\geq 1} j^\alpha p_{1j}$ *must be finite (anyway true for* $\alpha \leq 1$*).*

Now we first need to prove a lemma which is going to be used frequently in this and the next chapter. We define the following functions, where still $x^{(M)}$ is the Markov Process behaving according to SNM:

$$m_\alpha^M(t) := \sum_{j\geq 1} j^\alpha x_j^{(M)}(t), \text{ and } m_\alpha^\infty(t) := \sum_{j\geq 1} j^\alpha \xi_j(t),$$

$$c_\alpha(x) := \sum_{j\geq 1} j\mu x_j\{(j-1)^\alpha - j^\alpha\} + \lambda x_0 \sum_{k\geq 1}\sum_{j\geq 1} x_j p_{jk} k^\alpha - \kappa \sum_{j\geq 1} j^\alpha x_j,$$

$$W_\alpha^M(t) := m_\alpha^M(t) - m_\alpha^M(0) - \int_0^t c_\alpha(x^{(M)}(u))du.$$

Lemma 3.14 *For $\alpha \in (0,1]$ $W_\alpha^M(t)$ is an $\mathcal{H}_t$-martingale.*
If additionally y is such that $s_\alpha := \sum_{j\geq 1} j^\alpha y_j < \infty$ and ξ is a solution of DNM with $\xi(0) = y$, then

$$m_\alpha^\infty(t) = m_\alpha^\infty(0) + \int_0^t c_\alpha(\xi(u))du, \tag{3.9}$$

for $t \geq 0$.

Proof of Lemma 3.14 That W_α^M is an $\mathcal{H}_t$-martingale is proved in the Appendix as Corollary A10.

So we only have to prove equation (3.9). This happens in the next four steps:

1) Let $x^{(M)}$ be such that $x^{(M)}(0) \to y$ and $\sum_{j\geq 0} j^\alpha x_j^{(M)}(0) \to s_\alpha$. Then for $\beta < \alpha$ we have for arbitrary $J \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}[|m_\beta^M(t) - m_\beta^\infty(t)|] = \mathbb{E}[|\sum_{j\geq 0} j^\beta(x_j^{(M)}(t) - \xi_j(t))|] \\
&= \mathbb{E}[|\sum_{j<J} j^\beta(x_j^{(M)}(t) - \xi_j(t)) + \sum_{j\geq J} j^\beta(x_j^{(M)}(t) - \xi_j(t))|] \\
&\leq \mathbb{E}[|\sum_{j<J} j^\beta(x_j^{(M)}(t) - \xi_j(t))|] + \mathbb{E}[\sum_{j\geq J} j^\beta x_j^{(M)}(t)] + \sum_{j\geq J} j^\beta \xi_j(t) \\
&\leq \mathbb{E}[|\sum_{j<J} j^\beta(x_j^{(M)}(t) - \xi_j(t))|] + J^{\beta-\alpha}\mathbb{E}[m_\alpha^M(t)] + J^{\beta-\alpha}\sum_{j\geq 1} j^\alpha \xi_j(t).
\end{aligned} \tag{3.10}$$

2) Using Theorem 3.5 we can derive that

$$\mathbb{E}|[\sum_{j=0}^J j^\beta(x_j^{(M)}(t) - \xi_j(t))]| \to 0 \tag{3.11}$$

as $M \to \infty$ for all fix $J \geq 1$.

3) We are now going to prove that

$$\mathbb{E}[m_\alpha^M(t)] \leq m_\alpha^M(0)K(t), \tag{3.12}$$

where $K(t) < \infty$ for all $t \in (0, \infty)$. As W_α^M is a martingale, we have

$$\mathbb{E}[m_\alpha^M(t)] = m_\alpha^M(0) + \mathbb{E}[\int_0^t c_\alpha(x^{(M)}(u))du].$$

Looking at the definition of c_α we can derive

$$\begin{aligned}\mathbb{E}[m_\alpha^M(t)] &\leq m_\alpha^M(0) + \int_0^t \mathbb{E}[\lambda x_0^{(M)} \sum_{k\geq 1}\sum_{j\geq 1} x_j^{(M)} p_{jk} k^\alpha] du \\ &\leq m_\alpha^M(0) + \int_0^t \mathbb{E}[m_\alpha^M(u) K_1] du,\end{aligned}$$

where we used Lemma 3.13 in the last inequality and K_1 is some constant. Now we can apply the Gronwall-inequality which leads to (3.12).

4) Using (3.12) and the Lemma of Fatou we can derive that

$$m_\alpha^\infty \leq \limsup_{M\to\infty} \sum_{j\geq 0} j^\alpha \mathbb{E}[x_j^{(M)}(t)] \leq \lim_{M\to\infty} m_\alpha^M(0) K(t) \leq s_\alpha K(t).$$

Then, using (3.10), (3.11) and the result just above we see that

$$\limsup_{M\to\infty} \mathbb{E}|[m_\beta^M(t) - m_\beta^\infty(t)|] \leq J^{\beta-\alpha} s_\alpha K(t) + J^{\beta-\alpha} s_\alpha K(t)$$

and so, as $J \geq 1$ was arbitrary, we see that $\limsup_{M\to\infty} \mathbb{E}[|m_\beta^M(t) - m_\beta^\infty(t)|] = 0$, and so

$$\lim_{M\to\infty} \mathbb{E}[m_\beta^M(t)] = m_\beta^\infty(t). \tag{3.13}$$

Analogous calculations lead to

$$\lim_{M\to\infty} \mathbb{E}[\int_0^t c_\beta(x^{(M)}(u)) du] = \int_0^t c_\beta(\xi(u)) du. \tag{3.14}$$

Additionally, because again W_β^M is a martingale, we have

$$\mathbb{E}[m_\beta^M(t)] = m_\beta^M(0) + \mathbb{E}[\int_0^t c_\beta(x^{(M)}(u)) du]. \tag{3.15}$$

So from (3.13), (3.14) and (3.15) we have

$$m_\beta^\infty(t) = m_\beta^\infty(0) + \int_0^t c_\beta(\xi(u)) du$$

for all $\beta < \alpha$, and so by monotone convergence, letting β converge towards α we have (3.9) which finishes the proof of Lemma 3.14.

□

Proof of Theorem 3.12 This proof is similar to the proof of Theorem 3.2.

Without loss of generality let $\alpha \in (0,1)$ and choose $x^{(M)}(0) = y^M$ and y^M as in the statement of the theorem. Then fix $t > 0$ and $J \in \mathbb{N}$. As $\sum_{j\geq 0} \mathbb{E}[x_j^{(M)}(t)] = 1$, we have

$$|1 - \sum_{j<J} \mathbb{E}[x_j^{(M)}(t)]| = \sum_{j\geq J} \mathbb{E}[x_j^{(M)}(t)] \leq J^{-\alpha}\mathbb{E}[\sum_{j\geq J} j^\alpha x_j^{(M)}(t)]$$
$$\leq J^{-\alpha}\mathbb{E}[\sum_{j\geq 1} j^\alpha x_j^{(M)}(t)] = J^{-\alpha}\mathbb{E}[m_\alpha^M(t)],$$

and therefore using (3.12) we have

$$|1 - \sum_{j<J} \mathbb{E}[x_j^{(M)}(t)]| \leq J^{-\alpha} m_\alpha^M(0) K(t).$$

As all $x_j^{(M)}(t)$, $j \geq 0$, are uniformly integrable (even bounded!) and $x_j^{(M)}$ converges weakly to ξ_j we have $\mathbb{E}[x_j^{(M)}(t)] \to \mathbb{E}[\xi_j(t)]$ as M tends to infinity. So we can deduce that

$$|1 - \sum_{j<J} \xi_j(t)| \leq J^{-\alpha} s_\alpha K(t).$$

But as J was chosen arbitrarily we can let it go to ∞ and therefore we have $\sum_{j\geq 0} \xi_j(t) = 1$ which finishes the proof of the first part.

For the main part of Theorem 3.12 choose $\epsilon, \eta > 0$. Now pick J so large, that $J^{-\alpha} s_\alpha K(T) < \epsilon/4$ (note that $K(t)$ is monotonely increasing (see proof of Lemma 3.14)). By Theorem 3.5 we have that for each $j \geq 0$, $x_j^{(M)}$ converges weakly towards ξ_j in $D[0,T]$ and ξ_j is continuous. Therefore we can choose an M_0 such that

$$\mathbb{P}[\sup_{0\leq t\leq T} |x_j^{(M)}(t) - \xi_j(t)| > \frac{\epsilon}{4J}] \leq \frac{\eta}{J},$$

for all $M \geq M_0$, and $0 \leq j \leq J$. Define $A_j := \{\sup_{0\leq t\leq T} |x_j^{(M)}(t) - \xi_j(t)| < \epsilon/(4J)\}$, $A := \cap_{j<J} A_j$. Then for $\omega \in A$ we have for all $0 \leq t \leq T$

$$\sum_{j\geq 0} |x_j^{(M)}(\omega,t) - \xi_j(t)| \leq \sum_{j<J} |x_j^{(M)}(\omega,t) - \xi_j(t)| + \sum_{j\geq J} x_j^{(M)}(\omega,t) + \sum_{j\geq J} \xi_j(t)$$
$$\leq \frac{\epsilon}{4} + \sum_{j\geq J} x_j^{(M)}(\omega,t) + 1 - \sum_{j<J} \xi_j(t) \leq \frac{\epsilon}{4} + \sum_{j\geq J} x_j^{(M)}(\omega,t) + \frac{\epsilon}{4}$$
$$\leq \frac{\epsilon}{2} + (1 - \sum_{j<J} x_j^{(M)}(\omega,t)) = \frac{\epsilon}{2} + 1 + \sum_{j<J} (\xi_j(t) - x_j^{(M)}(\omega,t)) - \sum_{j<J} \xi_j(t)$$
$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + (1 - \sum_{j<J} \xi_j(t)) \leq \epsilon.$$

All that we need is to know the probability of the event A. But as

$$\mathbb{P}[\cup_{j<J} A_j^c] \leq \sum_{j<J} \mathbb{P}[A_j^c] \leq \sum_{j<J} \frac{\eta}{J} = \eta,$$

we therefore have $\mathbb{P}[A] \geq 1 - \eta$ which finishes the proof.

□

3.4 The linear models with mortality of humans

We again use the notation introduced in chapter 1. The two linear models with mortality of humans are models SLM and DLM. So we present how these two models are linked to each other. In Theorems 3.15 and 3.21 we work with X with transition rates as in SLM and Ξ behaving according to DLM. The first result shows that the (normalised) stochastic process SLM converges weakly towards the solution to the deterministic system DLM. We need sequences of processes behaving according to SL. We denote them with $(X^{(M)}, M \geq 1)$.

Theorem 3.15 *Let $(X^{(M)}, M \geq 1)$ be a sequence of Markov branching processes as specified in SLM. We assume that the initial state $X^{(M)}(0)$ is such that $\sum_{j\geq 1} X_j^{(M)}(0) < \infty$ and $M^{-1}X^{(M)}(0) \to y^{(0)}$, where $0 < \sum_{j\geq 1} jy_j^{(0)} < \infty$. Then $M^{-1}X^{(M)}$ converges weakly in $D^\infty[0, T]$ for each $T > 0$ to a non-random, nonnegative process Ξ, which evolves according to the differential equations DLM with initial state $\Xi(0) = y^{(0)}$, and satisfies conditions C.*

Remarks 1. That such a solution Ξ to DLM with given initial values exists and is unique is proved in Theorem 4.17. We do not use Theorem 3.15 to prove Theorem 4.17.

2. We can loosen the condition $0 < \sum_{j\geq 1} jy_j^{(0)} < \infty$ to $0 < \sum_{j\geq 1} y_j^{(0)} < \infty$. Then conditions C are not necessarily satisfied and all we can guarantee (of conditions C) is that $\sup_{0\leq s\leq t} \sum_{j\geq 1} \Xi_j(s) < \infty$ for all $t \geq 0$ and that there exists a $j \geq 1$ such that $\Xi_j(0) > 0$.

Proof of Theorem 3.15 The proof of Theorem 3.15 consists of several parts; additionally, we use Theorem 4.17. As the main ideas are the same as in the proof of Theorem 3.5, some explanations may be very brief.

For all $j \geq 1$ and $x \in \mathbb{R}_+^\infty$, define the functions

$$a_j(x) = (j+1)\mu x_{j+1} - j\mu x_j + \lambda \sum_{l\geq 1} x_l p_{lj} - \kappa x_j;$$

$$b_j(x) = (j+1)\mu x_{j+1} + j\mu x_j + \lambda \sum_{l\geq 1} x_l p_{lj} + \kappa x_j,$$

and the random processes

$$U_j^M(t) = x_j^M(t) - x_j^M(0) - \int_0^t a_j(x^M(u))du;$$
$$V_j^M(t) = U_j^M(t)^2 - \frac{1}{M}\int_0^t b_j(x^M(u))du,$$

where $x^M(t) := M^{-1}X^{(M)}(t)$. Further, let $\mathcal{I}_t^M$ denote $\sigma\{x^M(s),\ 0 \le s \le t\}$.

Lemma 3.16 *$U_j^M(t)$ and $V_j^M(t)$ are $\mathcal{I}_t^M$-martingales.*

Proof of Lemma 3.16 This lemma is proved in the Appendix as Corollary A11.

□

For any $T > 0$,

$$\mathbb{E}\Big\{\sup_{0\le t\le T}\sum_{l\ge 1} x_l^M(t)\Big\} \le e^{\lambda T}. \tag{3.16}$$

This last is true, because $\sum_{j\ge 1} X_j^{(M)}(t)$ only increases at an infection, and infections occur at a total rate of $\lambda\sum_{j\ge 1}\sum_{k\ge 1} X_j^{(M)} p_{jk} \le \lambda\sum_{j\ge 1} X_j^{(M)}$; thus, by comparison with a pure birth process with *per capita* birth rate λ, (3.16) follows.

Lemma 3.17 *For each j, the sequence $\{x_j^M\}_{M\ge 1}$ is tight in $D[0,T]$ and any weak limit belongs to $C[0,T]$.*

Proof of Lemma 3.17 We apply Billingsley (1968, Theorem 15.5), for which we need only to check that, given any $\epsilon, \eta > 0$, we can find $\delta, M_0 > 0$ such that, for all $M \ge M_0$,

$$\mathbb{P}[\sup_{0\le s<t\le T; t-s<\delta} |x_j^M(t) - x_j^M(s)| > \epsilon] < \eta. \tag{3.17}$$

From the definition of U_j^M, it follows that

$$\begin{aligned} |x_j^M(t) - x_j^M(s)| &\le |U_j^M(t) - U_j^M(s)| + \int_s^t |a_j(x^M(u))|du \\ &\le |U_j^M(t) - U_j^M(s)| + \int_s^t (\lambda\vee\mu\vee\kappa)2(j+1)\sum_{l\ge 1} x_l^M(u)du. \end{aligned} \tag{3.18}$$

Now, by the Doob-Kolmogorov inequality for martingales, for s arbitrary but fixed,

$$\begin{aligned}
\mathbb{P}\Big[\sup_{s<t\leq s+\delta}|U_j^M(t)-U_j^M(s)|\geq\frac{\epsilon}{6}\Big] &\leq \frac{36}{\epsilon^2}\mathbb{E}[(U_j^M(s+\delta)-U_j^M(s))^2]\\
&= \frac{36}{\epsilon^2 M}\mathbb{E}\Big[\int_s^{s+\delta} b_j(x^M(u))\,du\Big]\\
&\leq \frac{36}{\epsilon^2 M}\mathbb{E}\Big[\int_s^{s+\delta}(\lambda\vee\mu\vee\kappa)2(j+1)\sum_{l\geq 1}x_l^M(u)\,du\Big] \qquad (3.19)\\
&= \frac{72(j+1)(\lambda\vee\mu\vee\kappa)}{\epsilon^2 M}\int_s^{s+\delta}\mathbb{E}\Big[\sum_{l\geq 1}x_l^M(u)\Big]\,du\\
&\leq \frac{72(j+1)(\lambda\vee\mu\vee\kappa)\delta e^{\lambda T}}{\epsilon^2 M},
\end{aligned}$$

where the last inequality follows from (3.16). For the second term in (3.18), comparison with the pure birth process with rate λ immediately gives an estimate which is uniform in $s\in[0,T]$:

$$\begin{aligned}
\mathbb{P}\Big[\sup_{0\leq s<t\leq T;t-s\leq\delta}\int_s^t(\lambda\vee\mu\vee\kappa)2(j+1)\sum_{l\geq 1}x_l^M(u)du\geq\frac{\epsilon}{2}\Big]\\
\leq\frac{4\delta(j+1)(\lambda\vee\mu\vee\kappa)e^{\lambda T}}{\epsilon}. \qquad (3.20)
\end{aligned}$$

Hence, given $\epsilon,\eta>0$, pick δ so small that $4\delta(j+1)(\lambda\vee\mu\vee\kappa)e^{\lambda T}<\epsilon\eta/2$ with $\delta=T/r$ for some integer r, so that the estimate in (3.20) is at most $\eta/2$. We then choose M_0 so large that, for all $M\geq M_0$, $72(j+1)(\lambda\vee\mu\vee\kappa)e^{\lambda T}<\epsilon^2 M\eta/2T$, so that, from (3.19),

$$\frac{1}{\delta}\mathbb{P}[A_s^M]<\frac{\eta}{2T}$$

for any $s\in[0,T]$ and $M\geq M_0$, where $A_s^M:=\{\sup_{s<t\leq s+\delta}|U_j^M(t)-U_j^M(s)|\geq\epsilon/6\}$. With these choices, we have

$$\begin{aligned}
\mathbb{P}[\sup_{0\leq s<t\leq T;t-s<\delta}|U_j^M(t)-U_j^M(s)|>\epsilon/2] &\leq \mathbb{P}\Big[\bigcup_{i=0}^{r-1}A_{i\delta}^M\Big]\leq\sum_{i=0}^{r-1}\mathbb{P}[A_{i\delta}^M]\\
&\leq\frac{r\eta\delta}{2T}=\frac{\eta}{2}
\end{aligned}$$

for all $M\geq M_0$. This completes the proof of (3.17).

□

Lemma 3.18 *Given any infinite subsequence N_1 of $\mathbb{N}$, there exists a subsequence $N \subset N_1$ such that x^M converges weakly in $D^\infty[0,T]$ along N. We denote the limit by $x^* = x^*(N)$.*

Proof of Lemma 3.18 It is enough by Prohorov's theorem to show that the sequence x^M is tight in $D^\infty[0,T]$. Given $\epsilon > 0$, let K_j be a compact set in $D[0,T]$ such that $\mathbb{P}[x_j^M \in K_j] > 1 - 2^{-j}\epsilon$: such a K_j exists, by Lemma 3.17. Then $K = \prod_{j\geq 1} K_j$ is compact in $D^\infty[0,T]$, and $\mathbb{P}[x^M \in K] > 1 - \epsilon$.

□

Lemma 3.19 *$x^M \Rightarrow x^*(N)$ implies that $U_j^M \Rightarrow U_j^*$ in $D[0,T]$ along N, for any $j \geq 1$: here, $U_j^* = x^*(t) - x^*(0) - \int_0^t a_j(x^*(u))\,du$.*

Proof of Lemma 3.19 Define the functions

$$h(x)(t) := x_j(t) - x_j(0) - \int_0^t [(j+1)\mu x_{j+1}(u) - j\mu x_j(u) + \lambda \sum_{l\geq 1} x_l(u)p_{lj} - \kappa x_j(u)]\,du$$

and, for any $k > 0$,

$$h_k(x)(t) := x_j(t) - x_j(0) - \int_0^t [(j+1)\mu x_{j+1}(u) - j\mu x_j(u) + \lambda \sum_{l\geq 1} (x_l(u) \wedge k)p_{lj} - \kappa x_j(u)]\,du.$$

Then if $x \in D^\infty[0,T]$ satisfies $\sup_{0\leq t\leq T} \sum_{j\geq 1} |x_j(t)| < \infty$, both $h(x)$ and $h_k(x)$ are elements of $D[0,T]$, and $U_j^M(t) = h(x^M)(t)$ and $U_j^*(t) = h(x^*)(t)$. We thus need to prove that

$$\lim_{M\to\infty} \mathbb{E}[f(h(x^M))] = \mathbb{E}[f(h(x^*))] \tag{3.21}$$

for each $f \in C^b(D[0,T])$.

Observe that, for any such f and any $k > 0$, we have

$$\begin{aligned} |\mathbb{E}[f(h(x^M))] - \mathbb{E}[f(h(x^*))]| &\leq |\mathbb{E}[f(h(x^M))] - \mathbb{E}[f(h_k(x^M))]| \\ &+ |\mathbb{E}[f(h_k(x^M))] - \mathbb{E}[f(h_k(x^*))]| + |\mathbb{E}[f(h_k(x^*))] - \mathbb{E}[f(h(x^*))]|. \end{aligned} \tag{3.22}$$

For the first term in (3.22), it follows from (3.16) that

$$|\mathbb{E}[f(h(x^M))] - \mathbb{E}[f(h_k(x^M))]| \leq 2\|f\|\mathbb{P}\Big[\sup_{0\leq t\leq T} \sum_{j\geq 1} x_j^M(t) > k\Big] \leq 2\|f\|e^{\lambda T}/k.$$

A similar argument can be used for $|\mathbb{E}[f(h_k(x^*))] - \mathbb{E}[f(h(x^*))]|$, since

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\le t\le T}\sum_{j\ge 1} x_j^*(t) > k\Big] &= \lim_{J\to\infty}\mathbb{P}\Big[\sup_{0\le t\le T}\sum_{j=1}^{J} x_j^*(t) > k\Big] \\ &\le \lim_{J\to\infty}\liminf_{M\to\infty}\mathbb{P}\Big[\sup_{0\le t\le T}\sum_{j=1}^{J} x_j^M(t) > k\Big] \le e^{\lambda T}/k,\end{aligned} \tag{3.23}$$

because $x^M \Rightarrow x^*$ and from (3.16). Finally, h_k is continuous at all points of $C^\infty[0,T]$ since $\sum_{l\ge 1} p_{lj} < \infty$ for all $j \ge 0$ (Lemma 2.1) and $\mathbb{P}[x^* \in C^\infty[0,T]] = 1$ in view of Lemmas 3.17 and 3.18; thus $h_k(x^M) \Rightarrow h_k(x^*)$, and so the remaining term $|\mathbb{E}[f(h_k(x^M))] - \mathbb{E}[f(h_k(x^*))]|$ in (3.22) is small as $M \to \infty$. Because k was chosen arbitrarily, this proves (3.21).

□

Lemma 3.20 $x^M \Rightarrow x^*(N)$ *implies that*

$$\mathbb{P}\Big[\sup_{0\le t\le T}|U_j^M(t)| > \epsilon\Big] \to 0$$

along N*, for each* $j \ge 1$ *and* $\epsilon > 0$.

Proof of Lemma 3.20 By the Doob-Kolmogorov inequality applied to U_j^M and from the martingale property of V_j^M, it follows that

$$\begin{aligned}\mathbb{P}\Big[\sup_{0\le t\le T}|U_j^M(t)| > \epsilon\Big] &\le \frac{1}{\epsilon^2}\mathbb{E}[U_j^M(T)^2] \le \frac{1}{M\epsilon^2}\int_0^T \mathbb{E}[b_j(x^M(u))]\,du \\ &\le \frac{1}{M\epsilon^2}\int_0^T 2(j+1)(\lambda\vee\mu\vee\kappa)e^{\lambda T}\,du = \frac{2(j+1)(\lambda\vee\mu\vee\kappa)Te^{\lambda T}}{M\epsilon^2},\end{aligned}$$

which converges to 0 as $M \to \infty$.

□

As a consequence of Lemmas 3.19 and 3.20 we therefore have,

$$\mathbb{P}\Big[x_j^*(t) = x_j^*(0) + \int_0^t a_j(x^*(u))\,du \text{ for all } 0 \le t \le T\Big] = 1,$$

for any weak limit $x^* = x^*(N)$. Thus x^* satisfies DLM with $x(0) = y^{(0)}$, as required. By Theorem 4.17, there is only one solution of DLM that satisfies conditions C. It thus simply remains to be shown that any $x^*(N)$ satisfies conditions C; but this follows from (3.23). As the solution x^* to DLM is the weak limit of nonnegative stochastic processes, the solution x^* to DLM is nonnegative too (as required). This completes the proof of Theorem 3.15.

□

In the linear cases the solution to the deterministic model is the expectation of the stochastic model. In chapter 3.2 we proved the result without mortality of humans (Theorem 3.4), now we treat the case where $\kappa > 0$.

Theorem 3.21 *Let X be a Markov process with rates given in SLM and with initial state $X(0)$ satisfying $\sum_{j\geq 1} X(0) =: M < \infty$, and set $\Xi(0) := M^{-1}X(0)$. Then y defined by*

$$y_j(t) := M^{-1}\mathbb{E}\{X_j(t)|X(0) = M\Xi(0)\}$$

satisfies the differential equations DLM with $y(0) = \Xi(0)$, as well as conditions C.

Proof of Theorem 3.21 Let $m_{ij}(t) := \mathbb{E}[X_j(t)|X(0) = e_i]$. Then because individuals act independently we have

$$y_j(t) = \sum_{i\geq 1} y_i^{(0)} m_{ij}(t).$$

We first prove that conditions C are satisfied. All that has to be shown is that $\sum_{j\geq 1} y_j(t) < \infty$ for all t. This is true, because $\sum_{j\geq 1} X_j(t)$ only increases at an infection, and infections occur at a total rate of $\lambda \sum_{j\geq 1}\sum_{k\geq 1} X_j p_{jk} \leq \lambda \sum_{j\geq 1} X_j$; thus, by comparison with a pure birth process with *per capita* birth rate λ, we have

$$\sum_{j\geq 1} m_{ij}(t) \leq e^{\lambda t} < \infty$$

for all $i \geq 1$ and $t \geq 0$ and so

$$\sum_{j\geq 1} y_j(t) < \sum_{i\geq 1} y_i^{(0)} \sum_{j\geq 1} m_{ij}(t) < e^{\lambda t} < \infty$$

for all t and so conditions C are satisfied.

Let $\mathcal{I}_t$ denote the σ-algebra $\sigma(X(s), 0 \leq s \leq t)$ and $h > 0$. Then by the Markov property we have

$$\mathbb{E}[X_j(t+h)|\mathcal{I}_t] = \sum_{i\geq 1} X_i(t) m_{ij}(h),$$

and so

$$h^{-1}\{y_j(t+h) - y_j(t)\} = \sum_{i\neq j} y_j(t) h^{-1} m_{ij}(t) + y_j(t) h^{-1}(m_{jj}(h) - 1). \quad (3.24)$$

Now we use the branching structure of X and condition on the time and outcome of the first transition. If we start with only one person infected with i worms, the probability that nothing happens until time h is $e^{-(\lambda+i\mu+\kappa)h}$; the density function of the first outcome at time u is $(\lambda + i\mu + \kappa)e^{-(\lambda+i\mu+\kappa)u}$; in case of a first outcome at time u that outcome is a death of a worm with probability $i\mu/(\lambda+i\mu+\kappa)$ (proportional to its contribution to that outcome), a new infection with probability $\lambda/(\lambda + i\mu + \kappa)$ and finally the death of the carrier with probability $\kappa/(\lambda + i\mu + \kappa)$. So we have the following equation

$$\begin{aligned} m_{ij}(h) =&\delta_{ij}e^{-(\lambda+i\mu+\kappa)h} + \int_0^h e^{-(\lambda+i\mu+\kappa)u}[i\mu m_{i-1,j}(h-u) \\ &+ \lambda \sum_{l\geq 0} p_{il}\{m_{lj}(h-u) + m_{ij}(h-u)\} + \kappa m_{0j}(h-u)]du, \end{aligned}$$

with $m_{0j}(t)$ taken to be zero for all t. So we can proceed to

$$\begin{aligned} e^{(\lambda+i\mu+\kappa)h}m_{ij}(h) =&\delta_{ij} + \int_0^h e^{-(\lambda+i\mu+\kappa)v}[i\mu m_{i-1,j}(v) \\ &+ \lambda m_{ij}(v) + \lambda \sum_{l\geq 0} p_{il}m_{lj}(v)]dv; \end{aligned}$$

hence each m_{ij} is differentiable, and

$$m_{ij}(0) = \delta_{ij};\ m'_{ij}(0) = i\mu\delta_{i,j+1} - i\mu\delta_{ij} + \lambda p_{ij} - \kappa\delta_{ij}.$$

Again by comparison with the pure birth process, we have

$$0 \leq h^{-1}m_{ij}(h) \leq h^{-1}(e^{\lambda h} - 1) \leq \lambda e^{\lambda}$$

for all $h \leq 1$, uniformly for all $i\neq j$, and hence, since $\sum_{j\geq 1} y_j(t) < \infty$, dominated convergence as $h \to 0$ in (3.24) shows that

$$y'_j(t) = \sum_{i\neq j} y_i(t)\{i\mu\delta_{i,j+1} + \lambda p_{ij}\} + y_j(t)\{-j\mu + \lambda p_{jj} - \kappa\},$$

so that y satisfies DLM with $y(0) = y^{(0)}$.

□

Remark We assume that there is no analogue of Theorem 3.4 and Theorem 3.21 in the two non-linear cases (even if one fits the initial values appropriately): It might not be very easy to prove this in general but there is a simple heuristic reason why this would be very surprising: The reason lies in the nonlinearity: for the equation $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ to hold, the random variables must be independent or at least uncorrelated (by definition). Now, let us take x_0 to be the X and any x_l to be Y in the infection process (in the sum $\lambda x_0(t) \sum_{l \geq 1} x_l(t) p_{lj}$). The rate of infections in the Markov process is the above rate and if one replaces the x's by the symbol for the solutions to the differential equations (ξ) one gets the rate of infections of the deterministic process. But as the x_0 and x_l are clearly correlated this "replacing" is not justified by simply taking the expectation of the Markov process. Therefore in the non-linear cases the solutions to the differential equations are unlikely to be the expectations of the Markov processes. But if we take the expectation of the Markov processes and then let M tend to infinity, the limit is the solution to the differential equations because the markov processes converge weakly towards the solutions of the differential equations and each coordinate is bounded by 1.

3.5 Open questions

1. Looking at the theorems about weak convergence; what is the rate of convergence?

2. Having proved a law of large numbers, can we prove a central limit theorem? There are various possibilities: single co-ordinates, a finite combination of co-ordinates or even the entire process (a measure-valued process). For one single co-ordinate j such a result could be that there is a diffusion limit for

$$\sqrt{M}\big(x_j^{(M)}(t) - \xi_j(t)\big)_{0 \leq t \leq T}$$

as $M \to \infty$ if the initial values are suitable.