

4 The deterministic approach

All four deterministic models are analysed in what follows. Results about thresholds, stationary solutions, convergence and periodicity are presented and all results are compared with each other and with the results of the corresponding stochastic approach. Additionally we look at a process related to the deterministic model without mortality of humans where real time is replaced by generation number.

To begin with we make some general remarks:

In this thesis we have usually first treated the non-linear models without mortality of humans, then the linear models without mortality of humans, then the non-linear models with mortality of humans and finally the linear models with mortality of humans. In this chapter we first treat the two models without mortality of humans (sections 4.1 and 4.2). Then we treat the process related to the deterministic model without mortality of humans where real time is replaced by generation number (section 4.3). Thereafter we treat the *linear* model with mortality of humans (section 4.4) and the *non-linear* model with mortality of humans (section 4.5). This change of concept is due to the proofs. We use some results of the linear model in chapter 4.4 to prove results in the non-linear model in chapter 4.5.

We repeat from chapter 2 that by the expression **threshold behaviour** we *usually* denote general statements of the following type: If $R_0 > 1$ the epidemic develops in deterministic systems and if $R_0 < 1$ the epidemic dies out. As we have already seen in the stochastic approach, the situation is more complex in our models. Therefore we have to develop the threshold conditions for each model in every possible relevant combination of parameters. We want to point out a difference between the linear and the non-linear models: in the linear models the epidemic dies out when the relevant $R_i < 1$ while in the non-linear case it may be sufficient that $R_i \leq 1$ (see “4.6 Open questions” 4.6.1). Focusing on the linear case and having the relevant R_i equal exactly 1 the epidemic even tends to a stationary solution under certain parameter conditions.

We want to write down what we understand by the expression **stationary solution**. Call $\bar{\xi}(t)$ a stationary solution of say DNM, if $\bar{\xi}_j(t) \geq 0$ for all $j \geq 0$, $\sum_{j \geq 0} \bar{\xi}_j(t) = 1$, and putting $\xi = \bar{\xi}$ in the right hand side of DNM gives zero: the solution to DNM with $\xi(0) = \bar{\xi}$ is then $\xi(t) = \bar{\xi}$ for all t .

In the linear models we neglect the number of uninfected individuals. We suppose that in comparison to the infected individuals there is an infinite reservoir of susceptibles. Therefore there is no $\bar{\Xi}_0$ in the linear models. Further, in the linear models, every scalar multiple $\eta \bar{\Xi}$, $\eta > 0$, of a stationary solution

$\bar{\Xi}$ is a stationary solution too. It is also clear that in the non-linear models e_0 and in the linear models 0 are automatically stationary solutions. We call these stationary solutions *trivial* in comparison to the *nontrivial*. We mention these trivial solutions throughout, although they do not satisfy conditions C (see chapter 1) in the linear cases, because conditions C ask for at least one co-ordinate $j \geq 1$ such that $\Xi_j(0) > 0$. According to the threshold conditions, a necessary assumption for the existence of stationary solutions is that the relevant (depending on what value θ takes) $R_i = 1$ in the linear cases and the relevant $R_i \geq 1$ in the non-linear cases.

As we see in what follows, we can only prove results about the general behaviour of solutions. We do not know the explicit form of the solutions; we do not even know the explicit form of the stationary solutions (except in trivial cases). In reality we may assume that $\kappa < \mu$, meaning that the death rate of worms is larger than the death rate of their hosts. Assuming this we know about the stationary solutions in all four models that for each $j \geq 1$ the following inequality must hold:

$$\bar{\xi}_{j+1} < \bar{\xi}_j.$$

This result follows immediately from the differential equations as for example in model DLM via

$$\bar{\xi}_j = \frac{(j+1)\mu}{j\mu + \kappa} \bar{\xi}_{j+1} + \frac{\lambda}{j\mu + \kappa} \sum_{l \geq 1} \bar{\xi}_l p_{lj} > \bar{\xi}_{j+1}.$$

Besides the trivial solutions e_0 and 0 respectively we know the explicit form of the unique stationary solutions in the case where $p_{10} + p_{11} = 1$. Uniqueness is shown in theorems to follow; but very easily the reader can verify the following solutions, since $\theta = p_{11}$:

in case DN) $\bar{\xi}_0^{(0)} = (R_0^{(0)})^{-1}$, $\bar{\xi}_1^{(0)} = 1 - (R_0^{(0)})^{-1}$ and $\bar{\xi}_j^{(0)} = 0$ for all $j \geq 2$ if $R_0^{(0)} > 1$,

in case DL) $\bar{\Xi}_1^{(0)} = K$, where $K > 0$ and $\bar{\Xi}_j^{(0)} = 0$ for all $j \geq 2$ if $R_0^{(0)} = 1$,

in case DNM) $\bar{\xi}_0 = R_0^{-1}$, $\bar{\xi}_1 = 1 - R_0^{-1}$ and $\bar{\xi}_j = 0$ for all $j \geq 2$ if $R_0 > 1$,

in case DLM) $\bar{\Xi}_1 = K$, where $K > 0$ and $\bar{\Xi}_j = 0$ for all $j \geq 2$ if $R_0 = 1$.

We are not going to discuss various possible distances. All that must hold for us to talk about **convergence towards a stationary solution** $\bar{\xi}$ is that all components of the converging function $y(t)$ must satisfy $\lim_{t \rightarrow \infty} y_i(t) = \bar{\xi}_i$ for all $i \geq 1$ (convergence in \mathbb{R}^∞).

Periodic solutions are not considered in previous publications. But we can use results of Barbour, Heesterbeek and Luchsinger (1996) and the corresponding results with mortality of humans to exclude periodic solutions in the linear models (that is periodic solutions which are not stationary solutions). A solution ξ is periodic if there exists a $\tau > 0$ (the period) such that $\xi(t+\tau) = \xi(t)$ for all t . All stationary solutions are periodic in this sense. If we talk about periodic solutions in what follows, we always mean periodic solutions which are not stationary solutions.

4.1 The deterministic non-linear model without mortality of humans DN

Let $\xi^{(0)}$ be a solution of DN. In ‘‘Remarks on the basic reproduction ratios I’’ (see section 2.1) we defined the following combinations of parameters: $R_0^{(0)} := \lambda\theta/\mu$ and $R_1^{(0)} := \lambda e \log \theta/\mu$.

Intuitively it is clear, that if on average a parasite has less than one offspring under ideal conditions, that is if $R_0^{(0)} < 1$, then the epidemic must die out. That is precisely the following result, which is valid for all θ .

Theorem 4.1 [Barbour and Kafetzaki (1993), Theorem 4.1] *If $R_0^{(0)} < 1$ and if $\xi^{(0)}(0) = y$ is such that $s_1 := \sum_{j \geq 1} j y_j$ is finite, then $\lim_{t \rightarrow \infty} \xi^{(0)}(t) = e_0$ and $\sum_{j \geq 1} j \xi_j^{(0)}(t) \leq s_1 e^{-(\mu - \lambda\theta)t}$.*

Remark Comparing this theorem with Theorem 4.2, we see that the first statement about extinction is stated again in Theorem 4.2 in a more general context. So it is mainly the upper bound on the average number of parasites per individual that is relevant in this theorem. In the linear model DL (see equation (4.7)) we see that in fact that upper bound is the *exact* number of parasites at time t in DL. Note that in the non-linear models $\sum_{j \geq 1} j \xi_j(t)$ denotes the average number of parasites at time t per individual and not the total number of parasites in the system, because ξ stands for *proportions* of people.

Now we proceed to the threshold results. The following theorem includes one part of Theorem 4.1.

Theorem 4.2 *Let $\xi^{(0)}(0)$ be the initial conditions in DN such that $0 < \sum_{j \geq 0} j \xi_j^{(0)}(0) < \infty$. Then we have:*

Case 1) $\theta \leq e$: Then $\lim_{t \rightarrow \infty} \xi^{(0)}(t) = e_0$ if $R_0^{(0)} < 1$, and if $R_0^{(0)} > 1$ then $\xi^{(0)}(t) \not\rightarrow e_0$ as $t \rightarrow \infty$.

Case 2) $\theta > e$: Then $\lim_{t \rightarrow \infty} \xi^{(0)}(t) = e_0$ if $R_1^{(0)} < 1$, and if $R_1^{(0)} > 1$ then $\xi^{(0)}(t) \not\rightarrow e_0$ as $t \rightarrow \infty$.

Remarks 1. By the expression $\xi^{(0)}(t) \not\rightarrow e_0$ we include the possibility that the limit may not even exist in the theorem above and in what follows.

2. Theorem 4.2 is not contradictory to Theorem 4.1: There, in fact we did make a statement which holds for all θ but just one-sided about extinction and we anyway have the inequality $R_0^{(0)} \geq R_1^{(0)}$ for all $\theta > 0$.

3. The stochastic analogue of Theorem 4.2 is Theorem 2.3 but the reader should notice Theorem 2.2 too.

4. We do not know what happens if the relevant $R_i^{(0)} = 1$ (see “4.6 Open questions” 4.6.1.).

Proof of Theorem 4.2 If $\theta \leq e$ and $R_0^{(0)} < 1$ we can apply Theorem 4.1. The case $\theta > e$ and $R_1^{(0)} < 1$ was proved in Barbour and Kafetzaki (1993) as Theorem 4.6. So we only need to prove that in cases 1) and 2) the infection does not die out if the relevant R_i is larger than 1.

The strategy of the proof is as follows: In the non-linear model DN the contact rate λ is decreased to the effective contact rate $\lambda\xi_0^{(0)}(t)$. If the disease is near to extinction, $\xi_0^{(0)}$ must be almost 1. So the non-linear process $\xi^{(0)}$ is almost a linear process $\Xi^{(0)}$ behaving according to DL. But by Remark 1 of Theorem 4.8 we know that the linear process $\Xi^{(0)}$ does not die out under the conditions mentioned above (we do not use Theorem 4.2 to prove Theorem 4.8). So we must show that there exists a linear process $\Xi^{(0)}$ such that $\Xi_j^{(0)} \leq \xi_j^{(0)}$ for $j \geq 1$ at least until there is no danger for the process $\xi^{(0)}$ to die out.

We prove both cases ($\theta \leq e$ & $\theta > e$) in one. If we write “ $R_i^{(0)}$ ”, we mean $R_0^{(0)}$ if $\theta \leq e$ and $R_1^{(0)}$ if $\theta > e$. Let us define $N(t) := \sum_{j \geq 1} \xi_j^{(0)}(t)$. The expression $\xi^{(0)}(t) \not\rightarrow e_0$ means that there exists an $\epsilon > 0$ such that if at some time t_1 we have $N(t_1) < \epsilon$, then there exists a $t_2 > t_1$ such that $N(t_2) \geq \epsilon$. Without loss of generality we choose $t_1 = 0$ and ϵ such that $(1 - \epsilon)R_i^{(0)} > 1$. We therefore have to show that there exists a $T^* > 0$ such that $N(T^*) \geq \epsilon$. Let us define $\lambda' := \lambda(1 - \epsilon)$ and let $\Xi^{(0)}$ be a solution of DL with parameters (λ', θ, μ) . We choose the initial values such that $\Xi_j^{(0)}(0) = \xi_j^{(0)}(0)$ for all $j \geq 1$. Then we define $L(t) := \sum_{j \geq 1} \Xi_j^{(0)}(t)$ and $T := \inf\{t : L(t) \geq \epsilon\}$.

By Remark 1 of Theorem 4.8 we have $T < \infty$. Now if there exists a $v \in [0, T]$ such that $N(v) \geq \epsilon$ we can choose $T^* := v$ and nothing remains to be proved. Otherwise we have $N(t) < \epsilon$ for all $t \in [0, T]$. If we can show that for all $t \in [0, T]$ and $j \geq 1$,

$$\Xi_j^{(0)}(t) \leq \xi_j^{(0)}(t) \tag{4.1}$$

we have finished the proof.

We have $N(t) = 1 - \xi_0^{(0)}(t) < \epsilon$ for all $t \in [0, T]$. So we have $\lambda' = \lambda(1 - \epsilon) < \lambda\xi_0^{(0)}$ for all $t \in [0, T]$. Although intuitively we might expect that we therefore

can easily prove (4.1) by just comparing the two systems DN and DL with each other, such approaches seem difficult to carry through. We therefore look at stochastic processes $(x^{(M,0)})$ and $X^{(0)}$ where such a comparison is possible through the coupling method. Then we use Theorems 3.1 and 3.3 to finish the proof.

We now construct the two stochastic processes: the non-linear process $x^{(M,0)}$ and the linear process $X^{(0)}$. We define the process $x^{(M,0)}$ as in chapter 1, developing according to SN, where the initial values are to be suitably chosen later and $1/M \ll \epsilon$. For this we define a trivariate Markov process $(x^{(M,0)}(t), X^{(0)}(t), x^{(r)}(t))$. “ r ” stands for residual. In fact, each of the components in $(x^{(M,0)}(t), X^{(0)}(t), x^{(r)}(t))$ are themselves infinite dimensional: The first component is an infinite vector $(x_j^{(M,0)}(t))_{j \geq 0}$ where the co-ordinates take values in $\mathbb{Z}M^{-1} \cap [0, 1]$, the second component is an infinite vector $(X_k^{(0)}(t))_{k \geq 1}$ where the co-ordinates take values on the natural numbers and the third component is an infinite vector $(x_j^{(r)}(t))_{j \geq 0}$ where the co-ordinates take values in $\mathbb{Z}M^{-1} \cap [0, 1]$. We choose the initial values to be such that $x_0^{(M,0)}(0) = x_0^{(r)}(0)$, $x_j^{(M,0)}(0) = M^{-1}X_j^{(0)}(0)$ for $j \geq 1$ and $x_k^{(r)}(0) = 0$ for $k \geq 1$.

We want the trivariate Markov process to satisfy the following requirements \mathcal{R} .

Our aim is to construct $x^{(M,0)}$ and $x^{(r)}$ such that $x_j^{(M,0)} = M^{-1}X_j^{(0)} + x_j^{(r)}$ almost surely for $j \geq 1$ at least in the beginning (as long as $x_0^{(M,0)} > 1 - \epsilon$). Then we have $x_j^{(M,0)}(t) \geq M^{-1}X_j^{(0)}(t)$ for $j \geq 1$ too in the beginning. Additionally we want $x^{(M,0)}$ to behave according to SN and $X^{(0)}$ to behave according to SL.

We begin with $x_0^{(M,0)} > 1 - \epsilon$. Until $x_0^{(M,0)} \leq 1 - \epsilon$ for the first time, we let these processes develop according to the following rates:

$$(x^{(M,0)}, X^{(0)}, x^{(r)}) \rightarrow (x^{(M,0)} + M^{-1}(e_{j-1} - e_j), X^{(0)} + e_{j-1} - e_j, x^{(r)})$$

at rate $j\mu X_j^{(0)}$; $j \geq 2$, (death of a parasite in the linear process)

$$(x^{(M,0)}, X^{(0)}, x^{(r)}) \rightarrow (x^{(M,0)} + M^{-1}(e_0 - e_1), X^{(0)} - e_1, x^{(r)} + M^{-1}e_0)$$

at rate $\mu X_1^{(0)}$, (death of a parasite in an individual with only one parasite in the linear process)

$$(x^{(M,0)}, X^{(0)}, x^{(r)}) \rightarrow (x^{(M,0)} + M^{-1}(e_{j-1} - e_j), X^{(0)}, x^{(r)} + M^{-1}(e_{j-1} - e_j))$$

at rate $j\mu M x_j^{(r)}$; $j \geq 1$, (death of a parasite in the residual process)

$$(x^{(M,0)}, X^{(0)}, x^{(r)}) \rightarrow (x^{(M,0)} + M^{-1}(e_k - e_0), X^{(0)} + e_k, x^{(r)} - M^{-1}e_0)$$

at rate $\lambda' \sum_{u \geq 1} X_u^{(0)} p_{uk}$; $k \geq 1$, (infection in the linear process)

$$(x^{(M,0)}, X^{(0)}, x^{(r)}) \rightarrow (x^{(M,0)} + M^{-1}(e_k - e_0), X^{(0)}, x^{(r)} + M^{-1}(e_k - e_0))$$

at rate $\lambda x_0^{(M,0)} M \sum_{l \geq 1} x_l^{(r)} p_{lk} + (\lambda x_0^{(M,0)} - \lambda') \sum_{l \geq 1} X_l^{(0)} p_{lk}$, (infection in the residual process due to infective force of the residual process itself (first part of the rate) and due to residual rate (difference between the linear and non-linear contact rate, second part of the rate)). Note that $x_0^{(M,0)}(t) = x_0^{(r)}(t)$ until $x_0^{(M,0)} \leq 1 - \epsilon$ for the first time.

As soon as $x_0^{(M,0)}(t) \leq 1 - \epsilon$ for the first time, we let the linear process $X^{(0)}$ develop according to SL and independently of $x^{(M,0)}$.

The reader should notice that we have to distinguish carefully between the processes $x^{(M,0)}$ and $x^{(r)}$ on the one side and $X^{(0)}$ on the other side. The non-linear process and the residual process denote *proportions* of individuals while $X^{(0)}$ denotes the *explicit number*. This has to be considered while dealing with rates.

The reader can check that with our construction of the trivariate Markov process we meet all requirements \mathcal{R} .

We show (4.1) through contradiction: Suppose there is a $u \in [0, T]$ and a $J \in \mathbb{N} \setminus \{0\}$ such that

$$\Xi_J^{(0)}(u) > \xi_J^{(0)}(u). \quad (4.2)$$

$N = 1 - \xi_0^{(0)}$ and so N is a continuous function. Therefore there exists $q := \sup\{N(t) : t \in [0, T]\} < \epsilon$.

Now let us define $A_M := \{\omega : \sup_{0 \leq s \leq T} |x_0^{(M,0)}(s)(\omega) - \xi_0^{(0)}(s)| \leq \epsilon - q\}$. As by definition $q = \sup\{(1 - \xi_0^{(0)}(t)) : t \in [0, T]\}$, we have $A_M \subseteq \{\omega : x_0^{(M,0)}(t)(\omega) > (1 - \epsilon) \text{ for all } t \in [0, T]\}$. We now choose the initial values y^M of $x^{(M,0)}$ such that $y^M \rightarrow \xi^{(0)}(0)$ and $\sum_{j \geq 1} j y_j^M \rightarrow \sum_{j \geq 1} j \xi_j^{(0)}(0)$. By Theorem 3.2, $\mathbb{P}[A_M]$ converges to 1. We now define

$$B_M(u) := x_J^{(M,0)}(u) I_{A_M}, \quad C_M(u) := \frac{1}{M} X_J^{(0)}(u) I_{A_M}.$$

As $A_M \subseteq \{\omega : x_0^{(M,0)}(t)(\omega) > (1 - \epsilon), \text{ for all } t \in [0, T]\}$ we have by construction of the coupling $B_M(u) \geq C_M(u)$. But as M tends to ∞ , I_{A_M} converges weakly to 1, $x_J^{(M,0)}(u)$ converges weakly to $\xi_J^{(0)}(u)$ by Theorem 3.1 and $(1/M)X_J^{(0)}(u)$ converges weakly to $\Xi_J^{(0)}(u)$ by Theorem 3.3. But this is contradictory to (4.2) which finishes the proof. \square

Now we want to look at the average number of parasites at time t per individual. In comparison to the linear cases (see Remark 1 to Theorem 4.3), we do not have an explicit expression. But there are nevertheless some important remarks that can be made.

Theorem 4.3 *If $K := \sum_{j \geq 1} j \xi_j^{(0)}(0) < \infty$, then the average number of parasites per individual develops according to the following rule:*

$$\sum_{j \geq 1} j \xi_j^{(0)}(t) = K \exp\{(\lambda\theta - \mu)t - \lambda\theta \int_0^t \sum_{j \geq 1} \xi_j^{(0)}(u) du\}.$$

Remarks Some remarks about Theorem 4.3 follow after Theorem 4.25.

Proof of Theorem 4.3 The proof of Theorem 4.3 follows after the proof of Theorem 4.25. □

Let us now look at stationary solutions of DN. Call $\bar{\xi}^{(0)}(t)$ a stationary solution of DN, if for all $j \geq 0$ we have $\bar{\xi}_j^{(0)}(t) \geq 0$ and $\sum_{j \geq 0} \bar{\xi}_j^{(0)}(t) = 1$, and putting $\xi^{(0)} = \bar{\xi}^{(0)}$ in the right hand side of DN gives zero: the solution to DN with $\xi^{(0)}(0) = \bar{\xi}^{(0)}$ is then $\xi^{(0)}(t) = \bar{\xi}^{(0)}$ for all t .

Then, the results about stationary solutions are summarised in Theorems 4.4 and 4.5:

Theorem 4.4 [Barbour and Kafetzaki (1993), Theorems 4.2 and 4.6] *a) In DN we always have the trivial stationary solution $\bar{\xi}^{(0)} = e_0$ no matter which values the parameters take.*

b) There is no nontrivial stationary solution of DN with finite average number of parasites per individual if $\theta \geq e$.

c) Suppose that $\theta < e$ and $R_0^{(0)} > 1$. Then there exists a unique, nontrivial stationary solution of DN with finite average number of parasites per individual, though there may be other stationary solutions $\bar{\xi}^{(0)}$ for which $\sum_{j \geq 1} j \bar{\xi}_j^{(0*)} = \infty$. For the stationary solution $\bar{\xi}^{(0)}$ we furthermore have $\bar{\xi}_0^{(0)} = 1/R_0^{(0)}$.*

d) Assuming the conditions of c) and as long as $R_0^{(0)}$ remains greater than 1, the ratios $\bar{\xi}_j^{(0)}/(1 - \bar{\xi}_0^{(0)})$, $j \geq 1$, do not depend on λ or μ .

Remarks 1. We do not know anything about stationary solutions with infinite average number of parasites per individual (see “4.6 Open questions” 4.6.3).

2. We do not know whether in the situation of Theorem 4.4 c) a solution $\xi^{(0)}(t)$ converges towards $\bar{\xi}^{(0)}$ (see “4.6 Open questions” 4.6.2).

The next result is an important conclusion of Barbour and Kafetzaki (1993). It gives an idea of the possible dispersion in model DN. This result is discussed in chapter 5. It has an easily derived analogue in DL (Theorem 4.12 d)), but up until now it was not possible to expand these results to models DLM and DNm.

Theorem 4.5 [Barbour and Kafetzaki (1993), Theorem 4.4] *Suppose that $R_0^{(0)} > 1$ and that $\theta < e$, and let $\bar{\xi}^{(0)}$ be a stationary solution of DN with finite average number of parasites per individual. Then, for any $\alpha > 1$ the following statements hold:*

If $\theta \geq \alpha^{1/(\alpha-1)}$, then $\sum_{j \geq 1} j^\alpha \bar{\xi}_j^{(0)} = \infty$;

If $\theta < \alpha^{1/(\alpha-1)}$ and $\sum_{j \geq 1} j^\alpha p_{1j} < \infty$, then $\sum_{j \geq 1} j^\alpha \bar{\xi}_j^{(0)} < \infty$.

In the following corollary we add two results which help us to interpret the results in Theorem 4.5.

Corollary 4.6 *Suppose that $R_0^{(0)} > 1$ and that $\theta < e$, and let $\bar{\xi}^{(0)}$ be a stationary solution of DN with finite average number of parasites per individual. For $\alpha > 1$ and $\sum_{j \geq 1} j^\alpha p_{1j} < \infty$, fix θ and define α^* such that $\theta = \alpha^{*\frac{1}{\alpha^*-1}}$ and $f(\alpha) := \sum_{j \geq 1} j^\alpha \bar{\xi}_j^{(0)}$. Then for θ fixed we have*

a) $f(\alpha)$ is continuous and increasing in $\alpha \in (1, \alpha^*)$.

b)

$$\lim_{\alpha \nearrow \alpha^*} f(\alpha) = f(\alpha^*) = \infty.$$

Proof of Corollary 4.6 a) That f is increasing in $\alpha \in (1, \alpha^*)$ is clear. f is continuous in $\alpha \in (1, \alpha^*)$ because of monotone convergence.

b) This follows again through monotone convergence. □

The only result we found about convergence towards a nontrivial stationary solution of DN is

Theorem 4.7 [Barbour and Kafetzaki (1993), Theorem 4.6] *Let $\theta \geq e$ and $\bar{\xi}^{(0)}$ be a nontrivial stationary solution of DN. For a solution $\xi^{(0)}$ of DN with initial conditions $\xi^{(0)}(0) = y$ where $\sum_{j \geq 1} j y_j < \infty$ we have the following behaviour: If $R_1^{(0)} > 1$, then $\lim_{t \rightarrow \infty} \xi^{(0)}(t) = \bar{\xi}^{(0)}$ is only possible if $\bar{\xi}_0^{(0)} \geq 1/R_1^{(0)}$*

We have to add one remark about the border of the two regions of θ : If $\theta = e$, then we have $R_0^{(0)} = R_1^{(0)}$. So in Theorem 4.2 it does not matter whether we include the case $\theta = e$ in case 1) or 2). But comparing Theorem 4.4 b) and c) we see that for $\theta = e$ we have no nontrivial stationary solution of DN with finite average number of parasites per individual.

4.2 The deterministic linear model without mortality of humans DL

Let $\Xi^{(0)}$ be a solution to the system DL. In Theorem 4.8 we make a statement about the asymptotic behaviour of the number of infected individuals in DL and in Remark 1 following the theorem we derive the threshold result:

Theorem 4.8 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.6] *Assume that $\sum_{j \geq 1} j \Xi_j^{(0)}(0) < \infty$. Then the limit:*

$$\lim_{t \rightarrow \infty} t^{-1} \log \sum_{j \geq 1} \Xi_j^{(0)}(t) =: c^*(\lambda, \mu, \theta) =: c^*$$

exists, and is given by:

$$c^* = \begin{cases} \lambda\theta - \mu & \text{if } R_0^{(0)} \log \theta \leq 1 \\ \frac{\lambda\theta}{R_0^{(0)} \log \theta} (1 + \log(R_0^{(0)} \log \theta)) - \mu & \text{if } 1 < R_0^{(0)} \log \theta \leq \theta \\ \lambda & \text{if } R_0^{(0)} \log \theta > \theta. \end{cases}$$

Remarks 1. Simple calculations using Theorem 4.8 show that the following result holds too: If $\theta \leq e$, then $c^* < 0$ if and only if $R_0^{(0)} < 1$; if $\theta > e$, then $c^* < 0$ if and only if $(\lambda e \log \theta) / \mu < 1$.

2. The stochastic analogue of Remark 1 is Theorem 2.5.

The proofs of some theorems to come are based on **three main ideas**: **First**, the number of parasites in a stationary solution with finite number of parasites is constant. **Second**, the non-linear and the linear system can be compared and simple results about the number of non infected can be found. Additionally stationary solutions can be transferred from the non-linear systems to the linear systems and vice versa as is shown below. **Third**, the auxiliary Markov process in continuous time $Y(t)$ (see below) can be used. The classification into positive recurrence, null-recurrence and transience can be used for stationary solutions, convergence towards stationary solutions and periodic solutions.

We first have to define the auxiliary Markov process Y (see Barbour, Heesterbeek and Luchsinger (1996), chapter 2 for more detail). We first define the infinite dimensional matrix R as follows:

$$R_{ij} := i\mu(\delta_{i-1,j} - \delta_{ij}) + \lambda p_{ij}, \quad i, j \geq 1, \quad (4.3)$$

where δ_{kl} is the Kronecker delta. The equation $\dot{\Xi}^{(0)} = \Xi^{(0)}R$ is exactly our system DL. Then we define the matrix S as follows:

$$S_{ij} := \frac{j}{i}(R_{ij} - (\lambda\theta - \mu)\delta_{ij}), \quad i, j \geq 1. \quad (4.4)$$

This matrix has nonnegative off-diagonal elements and satisfies

$$\sum_{j \geq 1} S_{ij} = \frac{1}{i} \sum_{j \geq 1} jR_{ij} - (\lambda\theta - \mu) = 0$$

for all $i \geq 1$, and is thus a ‘‘Q-matrix’’ in the sense of Markov chain theory (see Chung (1967), page 134). Furthermore, if $y(t)$ is a nonnegative solution of $\dot{y} = yS$ with $y(0) = y^{(0)}$, then setting

$$\Xi_j^{(0)}(t) = j^{-1}y_j(t)e^{(\lambda\theta - \mu)t}, \quad t \geq 0, j \geq 1, \quad (4.5)$$

yields a nonnegative solution to $\dot{\Xi}^{(0)} = \Xi^{(0)}R$ with $\Xi_j^{(0)}(0) = j^{-1}y_j^{(0)}$, since

$$\begin{aligned} \dot{\Xi}^{(0)} &= j^{-1}(\dot{y}_j + (\lambda\theta - \mu)y_j)e^{(\lambda\theta - \mu)t} \\ &= j^{-1} \left[\sum_{i \geq 1} y_i S_{ij} + (\lambda\theta - \mu)y_j \right] e^{(\lambda\theta - \mu)t} \\ &= j^{-1} \sum_{i \geq 1} i^{-1} y_i j R_{ij} e^{(\lambda\theta - \mu)t} = \sum_{i \geq 1} x_i R_{ij}. \end{aligned}$$

Similarly, starting with a nonnegative solution of $\dot{\Xi}^{(0)} = \Xi^{(0)}R$, one obtains a solution of $\dot{y} = yS$ from (4.5). Thus the solutions of DL are simply related to the solution of $\dot{y} = yS$, for a specific Q-matrix S . This enabled us to prove the following theorem

Theorem 4.9 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.3] *The equations $\dot{\Xi}^{(0)} = \Xi^{(0)}R$ (the system DL), with $\Xi^{(0)}(0)$ such that $0 < \sum_{j \geq 1} \Xi_j^{(0)}(0) < \infty$, have a unique nonnegative solution satisfying $\sup_{0 \leq s \leq t} \sum_{j \geq 1} \Xi_j^{(0)}(s) < \infty$ for all $t \geq 0$. The solution is given by*

$$\Xi_j^{(0)}(t) = j^{-1} \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \mathbb{P}_l[Y(t) = j] \right) e^{(\lambda\theta - \mu)t},$$

where Y is the unique pure jump Markov process with Q-matrix S and \mathbb{P}_l denotes probability conditional on $Y(0) = l$.

Remark In conditions C we even assume that $\sum_{j \geq 1} j \Xi_j^{(0)}(0) < \infty$. Then the solution can be even more simply expressed as

$$\Xi_j^{(0)}(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \right) e^{(\lambda\theta - \mu)t} \mathbb{P}^0[Y(t) = j], \quad (4.6)$$

for all $j \geq 1$, where \mathbb{P}^0 denotes probability conditional on the initial distribution

$$\mathbb{P}^0[Y(0) = j] = j \Xi_j^{(0)}(0) / \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \right)$$

for Y . Looking at (4.6), we see that therefore the number of parasites must develop according to the following equation

$$\sum_{j \geq 1} j \Xi_j^{(0)}(t) = \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \right) e^{(\lambda\theta - \mu)t}. \quad (4.7)$$

We see that the number of parasites grows exactly when $R_0^{(0)}$ is larger than 1 no matter what values θ takes.

Remark on the auxiliary Markov process Y emerged purely as an analytical aid in (4.4). It also has a biological interpretation. Suppose, in an infinite population setting, that $q_i := q_i(t)$ denotes the proportion of parasites at time t which are living in hosts who have exactly i parasites; thus $q_i = i \Xi_i^{(0)} / \sum_{j \geq 1} j \Xi_j^{(0)}$. Using (4.7) and the definition of the system DL, it follows that

$$\dot{q}_i = i\mu q_{i+1} - [(i-1)\mu + \lambda\theta]q_i + \lambda \sum_{l \geq 1} q_l l^{-1} i p_{li}, \quad (4.8)$$

so that the $q_i(t)$ solve the Kolmogorov forward differential equations for the Markov process Y ; $q_i(t) = \mathbb{P}[Y(t) = i]$, with $\mathbb{P}[Y(0) = i]$ assigned according to the initial constitution of the population. Thus the distribution of $Y(t)$ is that of the number of parasites in a host which is selected at time t by choosing a *parasite* uniformly at random and then fixing on its host.

We could derive equation (4.7) from (4.6) because we multiplied the j -equation with j and added all equations for $j \geq 1$. Then in equation (4.7) the distribution of Y does not show up explicitly. But if one is interested in the number of infected individuals, the distribution (the term “ $\mathbb{P}^0[Y(t) = j]$ ”) still shows up. For proving Theorem 4.8 the classification of the Markov process Y into positive recurrence, null-recurrence and transience was therefore vital. This classification was proved in Theorem 4.10 which is as follows:

Theorem 4.10 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.5] *The Markov process Y is irreducible if $p_{10} + p_{11} < 1$. Assuming this, Y is positive recurrent if $R_0^{(0)} \log \theta < 1$, null recurrent if $R_0^{(0)} \log \theta = 1$ and transient if $R_0^{(0)} \log \theta > 1$. If $p_{10} + p_{11} = 1$, Y is eventually absorbed in state 1.*

Using Theorem 4.10 we state Lemma 4.11 which enables us to simplify many of the following proofs.

Lemma 4.11 *Suppose that in DL we have $R_0^{(0)} = 1$. The initial values are such that $0 < K := \sum_{l \geq 1} l \Xi_l^{(0)}(0) < \infty$. Then the following result holds:*

Case (1): $\theta < e$. Then there exists a unique infinite vector of positive real numbers v (the stationary distribution of Y under \mathbb{P}^0) such that $\sum_{j \geq 1} v_j = 1$ and

$$\lim_{t \rightarrow \infty} \Xi_j^{(0)}(t) = v_j K j^{-1} \quad \text{for all } j \geq 1.$$

Case (2): $\theta \geq e$. Then we have

$$\lim_{t \rightarrow \infty} \Xi_j^{(0)}(t) = 0 \quad \text{for all } j \geq 1.$$

Proof of Lemma 4.11 We can use equation (4.6), that is

$$\Xi_j^{(0)}(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \right) e^{(\lambda \theta - \mu)t} \mathbb{P}^0[Y(t) = j],$$

for all $j \geq 1$. As $R_0^{(0)} = 1$ (4.6) simplifies to

$$\Xi_j^{(0)}(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \right) \mathbb{P}^0[Y(t) = j],$$

for all $j \geq 1$. Looking at case (1), we have $\theta < e$. We can apply Theorem 4.10: If $p_{10} + p_{11} < 1$, we see that Y is positive recurrent because $R_0^{(0)} \log \theta < 1$. Therefore by general theory of Markov processes we have a unique infinite vector of positive real numbers v such that $\sum_{j \geq 1} v_j = 1$ and $\lim_{t \rightarrow \infty} \mathbb{P}^0[Y(t) = j] = v_j$ for all $j \geq 1$. If $p_{10} + p_{11} = 1$, Y is eventually absorbed in state 1. Then Lemma 4.11 is satisfied by choosing $v_1 = 1$.

Looking at case (2), we have $\theta \geq e$. We can apply Theorem 4.10 again: Here it is impossible that $p_{10} + p_{11} = 1$ because then $\theta \leq 1 < e$. So Y is either null recurrent or transient because $R_0^{(0)} \log \theta \geq 1$. But in both cases we have $\lim_{t \rightarrow \infty} \mathbb{P}^0[Y(t) = j] = 0$ for all $j \geq 1$. This ends the proof of Lemma 4.11. \square

Call $\bar{\Xi}^{(0)}(t)$ a stationary solution of DL, if $\bar{\Xi}_j^{(0)}(t) \geq 0$ for all $j \geq 0$, and putting $\Xi^{(0)} = \bar{\Xi}^{(0)}$ in the right hand side of DL gives zero: the solution to DL with $\Xi^{(0)}(0) = \bar{\Xi}^{(0)}$ is then $\Xi^{(0)}(t) = \bar{\Xi}^{(0)}$ for all t . The results concerning stationary solutions $\bar{\Xi}^{(0)}$ of the linear system DL are summarised in

Theorem 4.12 a) For every choice of parameters (λ, θ, μ) there exists the trivial stationary solution $\bar{\Xi}^{(0)} = 0$.

b) There is no nontrivial stationary solution of DL with finite number of parasites if $\theta \geq e$.

c) If $\theta < e$ and $R_0^{(0)} = 1$, then up to scalar multiplication there exists exactly one nontrivial stationary solution of DL with finite number of parasites.

d) Suppose that $R_0^{(0)} = 1$ and that $\theta < e$, and let $\bar{\Xi}^{(0)}$ be a stationary solution of DL with finite number of parasites. Then, for any $\alpha > 1$ the following statements hold:

$$\text{If } \theta \geq \alpha^{1/(\alpha-1)}, \text{ then } \sum_{j \geq 1} j^\alpha \bar{\Xi}_j^{(0)} = \infty;$$

$$\text{If } \theta < \alpha^{1/(\alpha-1)} \text{ and } \sum_{j \geq 1} j^\alpha p_{1j} < \infty, \text{ then } \sum_{j \geq 1} j^\alpha \bar{\Xi}_j^{(0)} < \infty.$$

e) Suppose that $R_0^{(0)} = 1$ and that $\theta < e$, and let $\bar{\Xi}^{(0)}$ be a stationary solution of DL with finite number of parasites. For $\alpha > 1$ and $\sum_{j \geq 1} j^\alpha p_{1j} < \infty$ fix θ and define α^* such that $\theta = \alpha^{*\frac{1}{\alpha^*-1}}$ and $f(\alpha) := \sum_{j \geq 1} j^\alpha \bar{\Xi}_j^{(0)}$. Then for θ fixed we have

e1) $f(\alpha)$ is continuous and increasing in $\alpha \in (1, \alpha^*)$.

e2)

$$\lim_{\alpha \nearrow \alpha^*} f(\alpha) = f(\alpha^*) = \infty.$$

Remarks 1. In Theorem 4.4 d) we saw that in the non-linear case the ratios $\bar{\xi}_j^{(0)}/(1 - \bar{\xi}_0^{(0)})$ are independent of λ and μ for $j \geq 1$ (as long as $R_0^{(0)}$ remains greater than 1). An analogous result in the linear case might be and in fact is that the ratios $\bar{\Xi}_i^{(0)}/\bar{\Xi}_j^{(0)}$ are independent of λ and μ for $i, j \geq 1$. But this result is obvious since in the linear case it is necessary that $R_0^{(0)} = 1$, i.e. $\lambda/\mu = 1/\theta$. So leaving θ constant and changing λ to $k\lambda$ for $k > 0$ requires also altering μ in the same manner and vice versa. All one is doing in such a case is changing the time scale.

2. We do not know anything about stationary solutions with infinite number of parasites or with infinite number of individuals (see “4.6 Open questions” 4.6.3. and 4.6.4.).

Proof of Theorem 4.12 a) is clear.

b) Suppose we have a nontrivial stationary solution of DL with finite number of parasites. As the number of parasites is finite and the solution is nontrivial, conditions C are satisfied. In a stationary solution the number of parasites is constant. By (4.7) this requires that $R_0^{(0)} = 1$. So we can apply Lemma 4.11, Case (2) which finishes the proof of Theorem 4.12 b).

c) A candidate for a nontrivial stationary solution with finite number of parasites can be found through Lemma 4.11 as follows: Suppose we have $\theta < e$, $R_0^{(0)} = 1$ and choose a fixed, finite number K (the (initial) number of parasites). Then for all $j \geq 1$ we define:

$$\bar{\Xi}_j^{(0)}(t) := j^{-1} K v_j \quad (4.9)$$

for all $t \geq 0$, where v denotes the unique stationary distribution of Y under $\mathbb{P}^{(0)}$. We now have to show that this is a solution of DL at all. Consider a solution y of DL with initial values as in (4.9). By Theorem 4.9 that solution y exists and is unique. We now have to show that the solution y is equal to (4.9) for all $t \geq 0$. Now y has a representation of the form (4.6). As $R_0^{(0)} = 1$ we have $\lambda\theta - \mu = 0$ and so we only have to ensure that if we start with $\mathbb{P}^{(0)}[Y(0) = j] = v_j$, then we have $\mathbb{P}^{(0)}[Y(t) = j] = v_j$ for all $t \geq 0$. But this is so because v is the stationary distribution of Y under $\mathbb{P}^{(0)}$.

Now we show that (4.9) is (up to scalar multiplication) the unique stationary solution of DL according to the way we defined such solutions. All co-ordinates of (4.9) are nonnegative (even positive). We have to show that if we put our solution (4.9) in the right side of DL we get zero. We therefore need

$$\mu v_{j+1} - \mu v_j + \lambda \sum_{i \geq 1} \frac{v_i}{i} p_{ij} = 0,$$

for all $j \geq 1$. By (4.4) we see that this is equivalent to $vS = 0$ as $\theta = \mu/\lambda$. But this is true as v is the unique stationary distribution of the Markov process associated with the Q-matrix S .

Uniqueness follows through contradiction. If z is an other stationary solution of DL and z is not a scalar multiple of $\bar{\Xi}^{(0)}$, we argue as follows: z must have a representation in the form of (4.6) too. We may assume without loss of generality that the initial total number of parasites is the same in z and $\bar{\Xi}^{(0)}$, because in the linear models, every scalar multiple of a stationary solution is a stationary solution too. But then, looking at (4.6) there must be two stationary distributions of Y , which is not possible.

d) We compare the stationary solutions of DL and DN with each other and make contradictions using Theorem 4.5. Choose an arbitrary $g_0 \in (0, 1)$.

If $\theta \geq \alpha^{1/(\alpha-1)}$, we assume we have a stationary solution $\bar{\Xi}^{(0)}$ of DL such that $\sum_{j \geq 1} j^\alpha \bar{\Xi}_j^{(0)} < \infty$. Without loss of generality we may choose that

$$\sum_{j \geq 1} \bar{\Xi}_j^{(0)} + g_0 = 1, \quad (4.10)$$

because in the linear models, every scalar multiple of a stationary solution is a stationary solution too. Now we define $\lambda' := \lambda/g_0$. As $R_0^{(0)} = 1$, we have $\lambda'\theta > \mu$. Therefore, by Theorem 4.4 c) we have a (unique), nontrivial stationary solution of DN $\bar{\xi}^{(0)}$ with parameters (λ', θ, μ) with finite average number of parasites per individual. Now we compare the two systems DN (with (λ', θ, μ)) and DL (with (λ, θ, μ)) with each other for $j \geq 1$. The two systems are:

$$\frac{d\xi_j^{(0)}}{dt} = (j+1)\mu\xi_{j+1}^{(0)} - j\mu\xi_j^{(0)} + \lambda'\xi_0^{(0)} \sum_{l \geq 1} \xi_l^{(0)} p_{lj}; \quad j \geq 1, \quad (\text{DN})$$

and

$$\frac{d\Xi_j^{(0)}}{dt} = (j+1)\mu\Xi_{j+1}^{(0)} - j\mu\Xi_j^{(0)} + \lambda \sum_{l \geq 1} \Xi_l^{(0)} p_{lj}; \quad j \geq 1. \quad (\text{DL})$$

The only difference between DN and DL is in the infection process: We have $\lambda'\xi_0^{(0)}$ in DN and only λ in DL. But by Theorem 4.4 c) we know that $\bar{\xi}_0^{(0)} = \mu/\lambda'\theta = \mu g_0/\lambda\theta = g_0$. Additionally we have by definition that $\lambda = \lambda'g_0$. So if we only want to look at stationary solutions (where $\xi_0^{(0)}$ is constant in time and equal to g_0), the two systems are in fact equal and both linear! We define: $g_j := \bar{\Xi}_j^{(0)}$ for $j \geq 1$. By (4.10) and the equivalence of DN and DL the vector $(g_j)_{j \geq 0}$ is the unique stationary solution of DN with parameters (λ', θ, μ) . So we constructed a stationary solution g of DN where for $\theta \geq \alpha^{1/(\alpha-1)}$ we have $\sum_{j \geq 1} j^\alpha g_j < \infty$. This is a contradiction to Theorem 4.5.

On the other hand, if $\theta < \alpha^{1/(\alpha-1)}$, we assume we have a solution $\bar{\Xi}^{(0)}$ of DL such that $\sum_{j \geq 1} j^\alpha \bar{\Xi}_j^{(0)} = \infty$. But then we can construct such a solution of DN too as shown above in the first part of d) which is again a contradiction to Theorem 4.5.

e) The proof of e) is the same as the proof of Corollary 4.6. □

In the next theorem we prove convergence of a solution of DL to a stationary solution under some obviously necessary assumptions.

Theorem 4.13 *Suppose that $\theta < e$ and $R_0^{(0)} = 1$. Then each solution z of DL, which satisfies conditions C converges towards the unique stationary solution $\bar{\Xi}^{(0)}$ of DL for which $\sum_{j \geq 1} j \bar{\Xi}_j^{(0)} = \sum_{j \geq 1} j z_j(0)$ holds.*

Proof of Theorem 4.13 We can use Lemma 4.11: we see that each solution of DL that satisfies conditions C converges to some infinite positive vector $(v_j K j^{-1})_{j \geq 1}$. We have to show (as in the proof of Theorem 4.12 c)) that this is the (unique) stationary solution of DL. We first have to show that this is a solution of DL at all. Consider a solution y of DL with initial values $(v_j K j^{-1})_{j \geq 1}$. By Theorem 4.9 that solution y exists and is unique. We now have to show that the solution y is equal to $(v_j K j^{-1})_{j \geq 1}$ for all $t \geq 0$. However, y has a representation of the form (4.6). As $R_0^{(0)} = 1$ we have $\lambda\theta - \mu = 0$ and so we only have to assure that if we start with $\mathbb{P}^{(0)}[Y(0) = j] = v_j$, then we have $\mathbb{P}^{(0)}[Y(t) = j] = v_j$ for all $t \geq 0$. But this is so because v is the stationary distribution of Y under $\mathbb{P}^{(0)}$. Now we show that $(v_j K j^{-1})_{j \geq 1}$ is a stationary solution of DL according to the way we defined such solutions. All co-ordinates of $(v_j K j^{-1})_{j \geq 1}$ are nonnegative (even positive). We have to show that if we put our solution $(v_j K j^{-1})_{j \geq 1}$ in the right side of DL we get zero. We therefore need

$$\mu v_{j+1} - \mu v_j + \lambda \sum_{i \geq 1} \frac{v_i}{i} p_{ij} = 0,$$

for all $j \geq 1$. By (4.4) we see that this is equivalent to $vS = 0$ as $\theta = \mu/\lambda$. But this is true as v is the unique stationary distribution of the Markov process associated with the Q-matrix S . Uniqueness follows from Theorem 4.12 c). \square

In the next theorem, which is valid for all θ , we rule out the possibility of periodic solutions in DL.

Theorem 4.14 *Apart from stationary solutions there are no periodic solutions of the linear system DL which satisfy conditions C.*

Proof of Theorem 4.14 In a periodic solution the number of parasites must be periodically the same too. But in view of (4.7) this means that $R_0^{(0)} = 1$ is necessary. Now we can apply Lemma 4.11: But the behaviour suggested in both cases rules out periodic solutions which are not stationary solutions. \square

Historical remarks on the threshold results in the models without mortality of humans SN, DN, SL, DL

Barbour and Kafetzaki (1993): Models SN and DN

In Barbour and Kafetzaki (1993) Theorem 2.2 (notation of this thesis) was proved, showing that in SN the epidemic dies out with probability 1 no matter what values the parameters take (the number of individuals is $M < \infty$). No stochastic threshold results (see Theorem 2.3) were found. In the deterministic

model DN they proved Theorem 4.1 which shows extinction for all θ as soon as $R_0^{(0)} < 1$. Then they proved the first part of Theorem 4.2 Case (2), that is: if $\theta > e$, then $\lim_{t \rightarrow \infty} \xi^{(0)}(t) = e_0$ if $R_1 < 1$. So they found (although only proving one direction) that either $R_0^{(0)}$ or $R_1^{(0)}$ could be the threshold for the extinction of the epidemic, depending on the value of θ .

Barbour (1994): Model SL

It was then in a stochastic environment and in a linear model (SL), that Barbour (1994) proved Theorem 2.5: it states that the threshold separating growth from extinction is $R_0^{(0)} = 1$ for $\theta < e$ and $R_1^{(0)} = 1$ for $\theta \geq e$. There are two objections to this result: First one can argue that this is only a linear approximation of the initial phase. Intuitively the case $R_i^{(0)} > 1$ is then not so clear, because in the non-linear case the contact rate is multiplied by the proportion of non infected. So it might be possible that the epidemic dies out nevertheless. On the other hand the case $R_i^{(0)} < 1$ is by the same reason even more evident. Looking at Theorem 4.2, we see that the above objection is not justified. On the other hand one might argue that in Barbour (1994) we are looking at a stochastic model. In a stochastic model an epidemic can die out by chance just at the beginning, even if the parameters are such that a substantial outbreak is well possible or even very likely. Thus a deterministic model may exhibit a different behaviour. So there is a need to look at a deterministic linear model too.

Barbour, Heesterbeek and Luchsinger (1996): Models SL and DL

Barbour, Heesterbeek and Luchsinger (1996) addressed this latter problem. Both thresholds $R_0^{(0)} = 1$ and $R_1^{(0)} = 1$ occur with same meaning (Remark 1 to Theorem 4.8); so the second objection is not justified either. In addition, it was possible to calculate the asymptotic growth rates (Theorem 4.8).

In this present thesis we have been able to show both sides of the threshold results in the non-linear cases, for all values of θ (Theorem 4.2).

4.3 A process related to DL where real time is replaced by generation number

Our discussion of model DL so far has been in terms of its evolution in real time. However, the basic reproduction number is usually defined by considering the reproductive success of an individual in terms of its offspring in the next generation; see, for example, Diekmann, Heesterbeek and Metz (1990). What

does the infection process look like, if it is described in terms of its evolution in time measured by generations of infected hosts?

As before, it is important to distinguish hosts according to their infective potential. Here, a type i host denotes a host who was *initially* infected by i parasites. The expected number T_{ij} of type j “offspring” in the next generation of infection arising from a single type i host is then given by

$$T_{ij} = \lambda \int_0^\infty \sum_{l=1}^i P_{il}(t) p_{lj} dt,$$

where $P_{il}(t)$ is the probability that, at time t after its infection, a type i host has exactly l surviving parasites: $P_{il}(t) = \mathbb{P}[Bi(i, e^{-\mu t}) = l]$. Since $\int_0^\infty P_{il}(t) dt = 1/l\mu$, the expected length of time for which exactly l parasites are alive in the host, we thus find that

$$T_{ij} = \frac{\lambda}{\mu} \sum_{l=1}^i l^{-1} p_{lj}.$$

Note that $\sum_{j \geq 1} j T_{ij} = i\lambda\theta/\mu = R_0^{(0)} i$; on average, parasite numbers are multiplied by $R_0^{(0)}$ in each generation.

The expected evolution, now with generation as the time parameter, is described by the discrete difference equations

$$x^{(n)} = x^{(n-1)} T,$$

where $x^{(n)}$ denotes the expected numbers of hosts of the different types in generation n . The operator T (or more precisely its adjoint) corresponds to the next-generation operator as introduced by Diekmann, Heesterbeek and Metz (1990). These equations are analogous to DL, with $T - I$ corresponding to R (defined in equation (4.3)), and have solution $x^{(n)} = x^{(0)} T^n$. To find its behaviour, note that the matrix L defined by

$$L_{ij} := \frac{j}{i R_0^{(0)}} T_{ij} = \frac{1}{i\theta} \sum_{l=1}^i l^{-1} j p_{lj} \quad (4.11)$$

is stochastic ($L - I$ corresponds to S), and that

$$T_{ij}^n = (R_0^{(0)})^n i L_{ij}^n / j.$$

Hence the development of $x^{(n)}$ depends on the properties of the Markov chain \hat{Y} with transition matrix L . In contrast to the evolution in real time, the matrix L does not involve λ or μ , so that critical values for determining the form of

the growth exponent of $x^{(n)}$ have to be different from those in Theorem 4.8. It turns out that the situation is rather simpler.

Theorem 4.15 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.8] *The Markov chain \hat{Y} with transition matrix L defined in (4.11) is positive recurrent if $\theta < e$, null recurrent if $\theta = e$ and transient if $\theta > e$. Furthermore, the limit*

$$\lim_{n \rightarrow \infty} \left\{ \sum_{j \geq 1} x_j^{(n)} \right\}^{1/n} =: \hat{c}(R_0^{(0)}, \theta)$$

exists, and is given by

$$\hat{c}(R_0^{(0)}, \theta) = \begin{cases} R_0^{(0)} & \text{if } \theta \leq e; \\ \lambda e \log \theta / \mu & \text{if } \theta > e. \end{cases}$$

The contrast between Theorems 4.8 and 4.15 indicates that the way in which generations of infection overlap in real time also changes in character, depending on the parameter values. As an aid to understanding this, let $q_{ik} := q_{ik}(t)$ denote the proportion of parasites at time t which are living in hosts who have exactly i parasites and belong to the k -th generation of infection. Then the analogue of (4.8) yields

$$\dot{q}_{ik} = i\mu q_{i+1,k} - [(i-1)\mu + \lambda\theta]q_{ik} + \lambda \sum_{l \geq 1} q_{l,k-1} l^{-1} i p_{li}.$$

Thus the $q_{ik}(t)$ solve the Kolmogorov forward equations for a two dimensional Markov process (Y, Z) with transitions

$$\begin{aligned} (j, k) &\rightarrow (j-1, k) \text{ at rate } (j-1)\mu; \\ (j, k) &\rightarrow (l, k+1) \text{ at rate } \lambda j^{-1} l p_{jl}, \end{aligned} \tag{4.12}$$

for $j, l \geq 1, k \geq 0$. Hence the distribution of $(Y(t), Z(t))$ can be interpreted as the status, in terms of number of parasites and generation, of a host selected at time t by choosing a parasite uniformly at random and then taking its host. The first component is the same as the original auxiliary Markov process Y . The second component Z is a Poisson process of rate $\lambda\theta$, showing that the proportion of parasites at time t which are living in generation k hosts is given by $e^{-\lambda\theta t} (\lambda\theta t)^k / k!$. Hence also, using (4.7), the proportion of the parasites of generation k which are alive at time t is given by $e^{-\mu t} (\mu t)^k / k!$. These are the same proportions as expected in a linear birth and death process with per

capita birth rate $\lambda\theta$ and death rate μ , suggesting that the overlap of generations in terms of parasite numbers remains entirely normal. However, host type and generation are not independent of one another: instead, we have the following result.

Theorem 4.16 [Barbour, Heesterbeek and Luchsinger (1996), Theorem 2.9] *For the Markov process (Y, Z) with transitions given in (4.12), we have*

$$\mathbb{E}[y^{Y(t)} z^{Z(t)} | Y(0) = i, Z(0) = 0] = e^{\lambda\theta t(z-1)} \mathbb{E}[y^{Y_z(t)} | Y_z(0) = i],$$

where Y_z is the auxiliary Y -process obtained when λ is replaced by λz . In particular, because the distribution of Y_z is not the same as that of Y , $Y(t)$ and $Z(t)$ are not independent.

Despite Theorem 4.16, it still makes sense to ask whether the distributions of $Y(t)$ and $Z(t)$ are in some sense asymptotically independent. If $R_0^{(0)} \log \theta < 1$, the Y -process is positive recurrent by Theorem 4.10, and so $Y(t)$ has a limiting distribution ν as $t \rightarrow \infty$. The distribution of $Z(t)$ never converges, but that of $\hat{Z}(t) := t^{-1/2}\{Z(t) - \lambda\theta t\}$ has limit $N(0, \lambda\theta)$. In the case when $R_0^{(0)} \log \theta < 1$, it follows from Theorem 4.16, by setting $z := \exp\{-st^{-1/2}\}$, that the pair $(Y(t), \hat{Z}(t))$ has $\nu \times N(0, \lambda\theta)$ as limiting distribution, with independence between the components.

Thus, sampling in real time, the distribution of host type settles asymptotically to a fixed distribution which is independent of generation number, whenever $R_0^{(0)} \log \theta < 1$. This remains true even when $\theta > e$, although, under these circumstances, the number of hosts in generation n decreases like $\{R_0^{(0)} e \log \theta / \theta\}^n$ instead of like $(R_0^{(0)})^n$, and so the average number of parasites per host at infection in generation n grows like $(\theta/e \log \theta)^n$. That these apparently different kinds of behaviour can coexist seems surprising. However, there are two factors which could help to account for it. First, even if the average number of parasites per host tends to infinity with generation, it need not be the case that the ‘‘typical’’ number of parasites per host, which is described by a probability distribution, also tends to infinity - a proper probability distribution can well have infinite mean. Secondly, a host with a large initial parasite load spends a longer time infected than one with a small initial load, but for most of this time the bulk of his initial parasites are already dead. However, from an epidemiological point of view, this combination of parameters is uninteresting, since then $R_0^{(0)} < 1$ also, and the infection dies out.

The more interesting case is that in which $R_0^{(0)} \log \theta > 1$ and $\theta < e$. This implies geometric growth of host and parasite numbers like $(R_0^{(0)})^n$ in terms of generations, but a slower exponential rate of growth in real time for the

number of infected hosts than the rate $e^{(\lambda\theta-\mu)t}$ for the number of parasites. Thus, in real time, the average number of parasites per host tends to infinity exponentially fast, although it remains steady when time is expressed in terms of number of generations. Once again, a proper distribution can have infinite mean; also, it is now not obviously the case from Theorem 4.16 that generation number and host type are asymptotically independent, so that the way in which the generations overlap may be biased by parasite burden. In both of these curious cases, the difficulties in interpreting the results arise because the hosts are sampled according to a scheme weighted by parasite number. Such a scheme has no meaning when the mean parasite burden is infinite, as may asymptotically be the case.

4.4 The deterministic linear model with mortality of humans DLM

As announced at the beginning of this chapter we first treat the linear case with mortality of humans and then in section 4.5 the non-linear case. This is due to the proofs: We need some results of 4.4 for to prove some results of 4.5.

Denote by Ξ a solution of the linear deterministic system of differential equations DLM, whilst still using $\Xi^{(0)}$ as a notation for a solution of DL. The system DLM is as follows:

$$\frac{d\Xi_j}{dt} = (j+1)\Xi_{j+1}\mu - j\Xi_j\mu + \lambda \sum_{l \geq 1} \Xi_l p_{lj} - \kappa \Xi_j, \quad j \geq 1. \quad (\text{DLM})$$

It can easily be verified that the solutions of DL and DLM are linked, in that the following equation is satisfied:

$$\Xi_j(t) = \Xi_j^{(0)}(t) e^{-\kappa t}, \quad (4.13)$$

for all $j \geq 1$. This should be understood in the sense that if one has a solution to either equation, one gets the solution to the other equation via formula (4.13). We know that a solution of DL is unique if $\sup_{0 \leq s \leq t} \sum_{j \geq 1} \Xi_j^{(0)}(s) < \infty$ for all $t \geq 0$ (Theorem 4.9). We now wish to look at the solutions of DLM which satisfy $\sup_{0 \leq s \leq t} \sum_{j \geq 1} \Xi_j(s) < \infty$ for all $t \geq 0$. We need to show that if we start with a solution of one of the systems and calculate a solution of the other system using (4.13), then this condition remains satisfied. Then we automatically have uniqueness of the solution of DLM with given initial values under the above condition. But, from (4.13), this condition is surely satisfied.

The following theorem is just a translation of Theorem 4.9 and the remark following it using relation (4.13). It shows that we have a unique nonnegative solution to system DLM.

Theorem 4.17 *The system DLM, with $\Xi(0)$ such that $0 < \sum_{j \geq 1} \Xi_j(0) < \infty$, have a unique nonnegative solution satisfying $\sup_{0 \leq s \leq t} \sum_{j \geq 1} \Xi_j(s) < \infty$ for all $t \geq 0$. The solution is given by*

$$\Xi_j(t) = j^{-1} \left(\sum_{l \geq 1} l \Xi_l(0) \mathbb{P}_l[Y(t) = j] \right) e^{(\lambda\theta - \mu - \kappa)t},$$

where Y is the unique pure jump Markov process with Q -matrix S (see 4.4) and \mathbb{P}_l denotes probability conditional on $Y(0) = l$.

Proof of Theorem 4.17 Theorem 4.17 is just a translation of Theorem 4.9 into the situation with mortality of humans. In view of remarks just before Theorem 4.17 nothing remains to be proved. \square

Remark In conditions C we even assume that $\sum_{j \geq 1} j \Xi_j(0) < \infty$. Then the solution can be even more simply expressed as

$$\Xi_j(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l(0) \right) e^{(\lambda\theta - \mu - \kappa)t} \mathbb{P}^0[Y(t) = j], \quad (4.14)$$

for all $j \geq 1$, where \mathbb{P}^0 denotes probability conditional on the initial distribution

$$\mathbb{P}^0[Y(0) = j] = j \Xi_j(0) / \left(\sum_{l \geq 1} l \Xi_l(0) \right)$$

Using (4.13) we gain two special solutions of the linear cases with and without mortality respectively: Starting with a nontrivial stationary solution $\Xi_j^{(0)}(t)$ in the model without mortality of humans DL and multiplying each co-ordinate with $e^{-\kappa t}$ we gain a solution of the model with mortality whose proportions amongst the co-ordinates stay constant while the whole system dies out. On the other hand, starting with a stationary solution of the system with mortality of humans DLM (such a stationary solution exists according to Theorem 4.20 c)) and multiplying with $e^{\kappa t}$, we gain a solution of the model without mortality where again the proportions amongst the co-ordinates stay constant over time; but this time we have growth in the system.

We can use equations (4.7) and (4.13) to compute the development of the number of parasites in the entire system:

$$\sum_{j \geq 1} j \Xi_j(t) = \left(\sum_{l \geq 1} l \Xi_l(0) \right) e^{(\lambda\theta - \mu - \kappa)t}. \quad (4.15)$$

Therefore $R_0 := \lambda\theta/(\mu + \kappa) = 1$ is the new threshold for the development of the number of parasites in DLM. Denote $R_0 := \lambda\theta/(\mu + \kappa)$, $R_1 := \lambda e \log \theta / (\mu \theta^{\frac{\kappa}{\mu}})$,

$R_2 := \lambda/\kappa$. The interpretations for these combinations of parameters have been given (as far as possible) in chapter 2 in “Remarks on the basic reproduction ratios I, II, III and IV”.

Additionally we gain a counterpart of Theorem 4.8 if we use (4.13): By looking at the definition of c^* in Theorem 4.8 and the definition of c^+ just below, we can derive that

$$c^+ = c^* - \kappa. \quad (4.16)$$

Using this we automatically have the following theorem. It makes a statement about the asymptotic behaviour of the number of infected individuals in DLM and in Remark 4 following the theorem we derive the threshold result:

Theorem 4.18 *Assume that $\sum_{j \geq 1} j \Xi_j(0) < \infty$. Then the limit:*

$$\lim_{t \rightarrow \infty} t^{-1} \log \sum_{j \geq 1} \Xi_j(t) =: c^+(\lambda, \mu, \theta, \kappa) =: c^+$$

exists and is given by:

$$c^+ = \begin{cases} \lambda\theta - \mu - \kappa & \text{if } R_0^{(0)} \log \theta \leq 1 \\ \frac{\lambda\theta}{R_0^{(0)} \log \theta} (1 + \log(R_0^{(0)} \log \theta)) - \mu - \kappa & \text{if } 1 < R_0^{(0)} \log \theta \leq \theta \\ \lambda - \kappa & \text{if } R_0^{(0)} \log \theta > \theta. \end{cases}$$

Remarks 1) This result is consistent with Theorem 4.8 if $\kappa = 0$.

2) One might ask oneself, why the death rate κ is so simply introduced in formula (4.15) and the calculation of c^+ . The reason is that the death rate is uniform (in our model) no matter how high the parasite burden is. So the proportions amongst the co-ordinates of Ξ stay unchanged. The death rate might influence the qualitative development of the disease; it can make a disease die out, i.e. make c^+ larger or smaller. But besides that κ only makes the disease spread at a more moderate speed ($c^+ > 0$) or increase the deterioration of the disease ($c^* < 0$ & $c^+ < 0$).

3) Here too, paradoxical behaviour as in models DN and DL is possible, that is: the number of parasites can tend to ∞ although the number of infected tends to 0. An example of such a combination of parameters is: $\lambda = 2, \theta = e, \mu = 1, \kappa = 2.5$. Then we have $R_0 > 1$ but $c^+ < 0$.

4) Using Theorem 4.18, elementary although quite complicated calculations lead to the following **threshold behaviour**: If we are only interested in whether $c^+ < 0$ or $c^+ > 0$ we have the following results:

In the region $\log \theta \leq (1 + (\kappa/\mu))^{-1}$ we have: $c^+ < 0$ if and only if $R_0 < 1$. In the region $(1 + (\kappa/\mu))^{-1} < \log \theta \leq \mu/\kappa$ we have: $c^+ < 0$ if and only if $R_1 < 1$. In the region $\log \theta > \mu/\kappa$ we have: $c^+ < 0$ if and only if $R_2 < 1$. This is consistent too with Remark 1 following Theorem 4.8 in case $\kappa = 0$; notice that these three regions for θ become two regions if κ tends to 0.

Proof of Theorem 4.18 Theorem 4.18 is just a translation of Theorem 4.8 into the situation with mortality of humans. In view of (4.16) nothing remains to be proved. \square

We now prove Lemma 4.19 which enables us to simplify many of the following proofs. The Markov process Y and the probability measure \mathbb{P}^0 have been defined in chapter 4.2 following Theorem 4.8.

Lemma 4.19 *Suppose that in DLM we have $R_0 = 1$. The initial values are such that $0 < K := \sum_{l \geq 1} l \Xi_l(0) < \infty$. Then the following result holds:*

Case (1): $\log \theta < 1/(1 + \kappa/\mu)$. Then there exists a unique infinite vector of positive real numbers v (the stationary distribution of Y under \mathbb{P}^0) such that $\sum_{j \geq 1} v_j = 1$ and

$$\lim_{t \rightarrow \infty} \Xi_j(t) = v_j K j^{-1} \quad \text{for all } j \geq 1.$$

Case (2): $\log \theta \geq 1/(1 + \kappa/\mu)$. Then we have

$$\lim_{t \rightarrow \infty} \Xi_j(t) = 0 \quad \text{for all } j \geq 1.$$

Proof of Lemma 4.19 We can use equation (4.6), that is

$$\Xi_j^{(0)}(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l^{(0)}(0) \right) e^{(\lambda\theta - \mu)t} \mathbb{P}^0[Y(t) = j],$$

for all $j \geq 1$. Using (4.13) we derive

$$\Xi_j(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l(0) \right) e^{(\lambda\theta - \mu - \kappa)t} \mathbb{P}^0[Y(t) = j], \quad (4.17)$$

for all $j \geq 1$. As $R_0 = 1$ (4.17) simplifies to

$$\Xi_j(t) = \frac{1}{j} \left(\sum_{l \geq 1} l \Xi_l(0) \right) \mathbb{P}^0[Y(t) = j],$$

for all $j \geq 1$. Looking at case (1), we have $\log \theta < 1/(1 + \kappa/\mu)$. We can apply Theorem 4.10 because we have

$$R_0^{(0)} \log \theta < \frac{\lambda\theta}{\mu} \frac{1}{1 + \kappa/\mu} = \frac{\lambda\theta}{\mu + \kappa} = 1.$$

So if $p_{10} + p_{11} < 1$, Y is therefore positive recurrent. Therefore by general theory of Markov processes we have a unique infinite vector of positive real numbers v such that $\sum_{j \geq 1} v_j = 1$ and $\lim_{t \rightarrow \infty} \mathbb{P}^0[Y(t) = j] = v_j$ for all $j \geq 1$. If $p_{10} + p_{11} = 1$, Y is eventually absorbed in state 1. Then Lemma 4.19 is satisfied by choosing $v_1 = 1$.

Looking at case (2), we have $\log \theta \geq 1/(1 + \kappa/\mu)$. We can apply Theorem 4.10 again: Here it is impossible that $p_{10} + p_{11} = 1$ because then $\log \theta > 1/(1 + \kappa/\mu) > 0$ and $p_{11} = \theta < 1$ in that case. Y is either null recurrent or transient because

$$R_0^{(0)} \log \theta \geq \frac{\lambda \theta}{\mu} \frac{1}{1 + \kappa/\mu} = \frac{\lambda \theta}{\mu + \kappa} = 1.$$

But in both cases we have $\lim_{t \rightarrow \infty} \mathbb{P}^0[Y(t) = j] = 0$ for all $j \geq 1$. This ends the proof of Lemma 4.19. □

The results about stationary solutions in DLM are summarised in the following theorem:

Theorem 4.20 *a) For every choice of parameters $(\lambda, \theta, \mu, \kappa)$ there exists the trivial stationary solution $\bar{\Xi} = 0$.*

b) There is no nontrivial stationary solution of DLM with finite number of parasites if $\log \theta \geq (1 + \kappa/\mu)^{-1}$.

c) If $\log \theta < (1 + \kappa/\mu)^{-1}$ and $R_0 = 1$, then up to scalar multiplication there exists exactly one nontrivial stationary solution of DLM with finite number of parasites.

Remarks 1. In Theorem 4.26 d) we see that in the non-linear case the ratios $\bar{\xi}_j/(1 - \bar{\xi}_0)$ are independent of λ, μ, κ for $j \geq 1$ (as long as R_0 remains greater than 1 and the ratio κ/μ is not altered). An analogous result in the linear case might be and in fact *is* that the ratios $\bar{\Xi}_i/\bar{\Xi}_j$ are independent of λ, μ, κ for $i, j \geq 1$ under the above constraints. But this result is obvious since in the linear case it is necessary that $R_0 = 1$, i.e. $\lambda/(\mu + \kappa) = 1/\theta$. So leaving θ constant and changing λ to $k\lambda$ for $k > 0$ requires also altering (μ, κ) in the same manner and vice versa. All one is doing in such a case is changing the time scale! Exactly the same calculations as in the proof of Theorem 4.26 d) show additionally that here too if the ratio κ/μ should be altered then not *all* above ratios can stay the same!

2. We do not know anything about stationary solutions with infinite number of parasites or with infinite number of individuals (see “4.6 Open questions” 4.6.3. and 4.6.4.).

Proof of Theorem 4.20 a) is clear.

b) The proof of Theorem 4.20 b) is almost the same as the proof of Theorem 4.12 b). Use equation (4.15) instead of (4.7) and apply Lemma 4.19, Case (2) instead of Lemma 4.11, Case (2).

c) The proof of Theorem 4.20 c) is almost the same as the proof of Theorem 4.12 c). Use Lemma 4.19 instead of Lemma 4.11, Theorem 4.17 instead of Theorem 4.9, equation (4.14) instead of (4.6), use $\theta = (\mu + \kappa)/\lambda$. \square

In the next theorem we prove convergence of a solution of DLM to a stationary solution under some obviously necessary assumptions.

Theorem 4.21 *If $\log \theta < (1 + \kappa/\mu)^{-1}$ and if $R_0 = 1$ then every solution y of DLM which satisfies conditions C converges towards that unique stationary solution $\bar{\Xi}$ of DLM which satisfies $\sum_{j \geq 1} j \bar{\Xi}_j = \sum_{j \geq 1} j y_j(0)$.*

Proof of Theorem 4.21 The proof of Theorem 4.21 is almost the same as the proof of Theorem 4.13. Use Lemma 4.19 instead of Lemma 4.11, Theorem 4.17 instead of Theorem 4.9, equation (4.14) instead of (4.6), use $\theta = (\mu + \kappa)/\lambda$, Theorem 4.20 c) instead of Theorem 4.12 c). \square

In the next theorem, which is valid for all θ , we rule out the possibility of periodic solutions in DLM.

Theorem 4.22 *In the linear system DLM there are no periodic solutions which satisfy conditions C except stationary solutions.*

Proof of Theorem 4.22 In a periodic solution the number of parasites must be periodically the same too. But in view of (4.14) this means that $R_0 = 1$ is necessary. Now we can apply Lemma 4.19: But the behaviour suggested in both cases rules out periodic solutions which are not stationary solutions. \square

4.5 The deterministic non-linear model with mortality of humans DNM

As promised, after having examined the linear model with mortality of humans, we now treat the non-linear case.

We are now going to look at the counterpart of Theorem 4.1. Again, it is valid for all values of θ . This time we are going to examine the non-linear, deterministic system DNM. So let ξ be the unique solution of DNM (with given initial values, see Theorem 3.5). Intuitively it is clear, that if a parasite has less than one offspring under ideal conditions, that is if $R_0 < 1$, then the epidemic must die out. That is precisely the following result:

Theorem 4.23 *If $R_0 < 1$ and if $\xi(0) = y$ is such that $s_1 := \sum_{j \geq 1} j y_j < \infty$, then $\lim_{t \rightarrow \infty} \xi(t) = e_0$ and $\sum_{j \geq 1} j \xi_j(t) \leq s_1 e^{-(\mu + \kappa - \lambda \theta)t}$.*

Proof of Theorem 4.23 We use the notation introduced in the proof of Theorem 3.12. We can use equation (3.9) from chapter 3 for $\alpha = 1$. So we have:

$$m_1^\infty(t) = m_1^\infty(0) + \int_0^t c_1(\xi(u)) du; \quad (4.18)$$

where $m_1^\infty(t) := \sum_{j \geq 1} j \xi_j(t)$. We have:

$$m_1^\infty(0) = \sum_{j \geq 1} j \xi_j(0) = \sum_{j \geq 1} j y_j = s_1,$$

and further by defining $\xi := \xi(t)$:

$$\begin{aligned} c_1(\xi) &= \sum_{j \geq 1} j \mu \xi_j [(j-1) - j] + \lambda \xi_0 \sum_{k \geq 1} \sum_{j \geq 1} \xi_j p_{jk} k - \kappa \sum_{j \geq 1} j \xi_j \\ &= - \sum_{j \geq 1} j \mu \xi_j + \lambda \xi_0 \sum_{k \geq 1} \sum_{j \geq 1} \xi_j p_{jk} k - \kappa \sum_{j \geq 1} j \xi_j \\ &= -(\mu + \kappa) \sum_{j \geq 1} j \xi_j + \lambda \xi_0 \sum_{k \geq 1} \sum_{j \geq 1} \xi_j p_{jk} k \\ &= -(\mu + \kappa) \sum_{j \geq 1} j \xi_j + \lambda \xi_0 \sum_{j \geq 1} \xi_j j \theta = (\lambda \xi_0 \theta - \mu - \kappa) \sum_{j \geq 1} j \xi_j. \end{aligned}$$

Therefore using (4.18) it follows that:

$$m_1^\infty(t) = s_1 + \int_0^t [\lambda \xi_0(u) \theta - \mu - \kappa] m_1^\infty(u) du.$$

Taking the derivative we get

$$\frac{dm_1^\infty(t)}{dt} - [\lambda \xi_0(t) \theta - \mu - \kappa] m_1^\infty(t) = 0. \quad (4.19)$$

But (4.19) is a linear differential equation with solution

$$m_1^\infty(t) = s_1 e^{\int_0^t [\lambda \xi_0(u) \theta - \mu - \kappa] du}, \quad (4.20)$$

because the initial conditions are $m_1^\infty(0) = s_1$. Because $\xi_0(u) \leq 1$ for all u , we have

$$m_1^\infty(t) \leq s_1 e^{-(\mu + \kappa - \lambda \theta)t}.$$

As $m_1^\infty(t) = \sum_{j \geq 1} j \xi_j(t)$, Theorem 4.23 is proved. \square

Next we derive the threshold results for system DNM:

Theorem 4.24 *Let $\xi(0)$ in DNM be such that $0 < \sum_{j \geq 0} j \xi_j(0) < \infty$.*

Then the following statements hold:

Case 1) $\log \theta \leq (1 + \kappa/\mu)^{-1}$: Then $\lim_{t \rightarrow \infty} \xi(t) = e_0$ if $R_0 < 1$, and if $R_0 > 1$ then $\xi(t) \not\rightarrow e_0$ as $t \rightarrow \infty$.

Case 2) $(1 + \kappa/\mu)^{-1} < \log \theta \leq \mu/\kappa$: Then $\lim_{t \rightarrow \infty} \xi(t) = e_0$ if $R_1 < 1$, and if $R_1 > 1$ then $\xi(t) \not\rightarrow e_0$ as $t \rightarrow \infty$.

Case 3) $\log \theta > \mu/\kappa$: Then $\lim_{t \rightarrow \infty} \xi(t) = e_0$ if $R_2 < 1$, and if $R_2 > 1$ then $\xi(t) \not\rightarrow e_0$ as $t \rightarrow \infty$.

Remarks 1. Theorem 4.24 is not contradictory to Theorem 4.23: There, in fact we did make a statement which holds for all θ but just one-sided about extinction and obviously we anyway have the inequalities $R_0 \geq R_1$ for all $\theta > 0$ and $R_0 \geq R_2$ for all θ such that $\log \theta > \mu/\kappa$.

2. The stochastic analogue of Theorem 4.24 is Theorem 2.8 but the reader should notice Theorem 2.7 too.

3. We do not know what happens if the relevant $R_i = 1$ (see “4.6 Open questions” 4.6.1.).

Proof of Theorem 4.24 The first case where $R_0 < 1$ follows immediately from Theorem 4.23. We first prove the results where the disease dies out (the relevant R_i must be smaller than 1). Then we prove that the infection does not die out if the relevant R_i is larger than one.

We use the notation of chapter 3 for c_α and $m_\alpha^\infty(t)$. The function $f(x) = x^\alpha$ is concave if $\alpha \in [0, 1]$. So for x_1, x_2 we have

$$f(x_1) \leq f(x_2) + f'(x_2)(x_1 - x_2).$$

If we choose $x_1 = j - 1$, $x_2 = j$ we therefore get

$$\{(j - 1)^\alpha - j^\alpha\} \leq -\alpha j^{\alpha-1},$$

and so we can derive

$$\sum_{j \geq 1} j \mu \xi_j \{(j - 1)^\alpha - j^\alpha\} \leq \mu \sum_{j \geq 1} j \xi_j (-\alpha j^{\alpha-1}) \leq -\mu \alpha \sum_{j \geq 1} j^\alpha \xi_j.$$

Using Jensen’s inequality for concave functions we have $\sum_{l \geq 0} p_{jl} l^\alpha \leq (j\theta)^\alpha$. So

$$\begin{aligned} \lambda \xi_0 \sum_{l \geq 1} \sum_{j \geq 1} \xi_j p_{jl} l^\alpha &= \lambda \xi_0 \sum_{j \geq 1} \xi_j \sum_{l \geq 1} p_{jl} l^\alpha \\ &\leq \lambda \theta^\alpha \xi_0 \sum_{j \leq 1} j^\alpha \xi_j, \end{aligned}$$

and so looking at the definition of c_α in chapter 3 we can conclude

$$\begin{aligned} c_\alpha(\xi) &= \sum_{j \geq 1} j \mu \xi_j \{(j-1)^\alpha - j^\alpha\} \\ &\quad + \lambda \xi_0 \sum_{l \geq 1} \sum_{j \geq 1} \xi_j p_{jl} l^\alpha - \kappa \sum_{j \geq 1} j^\alpha \xi_j \\ &\leq (\lambda \theta^\alpha \xi_0 - \mu \alpha - \kappa) \sum_{j \geq 1} j^\alpha \xi_j. \end{aligned}$$

Using (3.9) of chapter 3 we therefore have for $0 \leq v \leq t$

$$\begin{aligned} m_\alpha^\infty(t) &= m_\alpha^\infty(v) + \int_v^t c_\alpha(\xi(u)) du \\ &\leq m_\alpha^\infty(v) - \int_v^t (\mu \alpha + \kappa - \lambda \theta^\alpha \xi_0) \sum_{j \geq 1} j^\alpha \xi_j(u) du \quad (4.21) \\ &= m_\alpha^\infty(v) - \int_v^t (\mu \alpha + \kappa - \lambda \theta^\alpha \xi_0) m_\alpha^\infty(u) du. \end{aligned}$$

If $R_0 < 1$ we can use Theorem 4.23 which is valid for all θ and so we may assume that $R_0 \geq 1$. We want to find an $\alpha \in (0, 1]$ such that $g(\alpha) := \kappa + \mu \alpha - \lambda \theta^\alpha > 0$. Then we can apply the Gronwall-inequality to (4.21) because then $\lim_{t \rightarrow \infty} m_\alpha^\infty(t) = 0$ which ends the proof. Let us first analyse this function g : $g(0) = \kappa - \lambda$ and $g(1) = \kappa + \mu - \lambda \theta \leq 0$ because $R_0 \geq 1$. So if $\kappa - \lambda > 0$ (that is $R_2 < 1$) we easily find an $\alpha \in (0, 1]$ such that $g(\alpha) > 0$ ($\alpha \rightarrow 0$ finally gives us such an α). So the third case where $R_2 < 1$ is already satisfied. We may therefore assume that $\kappa \leq \lambda$. We therefore only have to show the second case: $1/(1 + \kappa/\mu) < \log \theta \leq \mu/\kappa$, $R_1 < 1$. Elementary calculations show that g takes the maximum with respect to α at

$$\alpha_0 := \frac{1}{\log \theta} \log \left(\frac{\mu}{\lambda \log \theta} \right).$$

Under the assumptions above it can be shown through elementary though partly tedious calculations that $\alpha_0 \in (0, 1]$ and $g(\alpha_0) > 0$. This ends the proof of the first directions ($R_i < 1$).

Now we need to prove that in cases 1), 2) and 3) the infection does not die out if the relevant R_i is larger than 1. This proof is almost the same as the proof of the second part of the proof of Theorem 4.2. The only major change is the introduction of mortality of individuals but that does not cause any problems. The main problem of these proofs lies in the infection process. The necessary changes are: use Remark 4 of Theorem 4.18 instead of Remark 1 of Theorem 4.8, use Theorems 3.5 and 3.15 instead of Theorems 3.1 and 3.3, use Theorem 3.12 instead of Theorem 3.2. □

Now we want to look at the average number of parasites per individual at time t . In comparison to the linear cases (see remarks), we do not have an explicit expression. But there are nevertheless some important remarks that can be made.

Theorem 4.25 *If $K := \sum_{j \geq 1} j \xi_j(0) < \infty$, then the average number of parasites per individual develops according to the following rule:*

$$\sum_{j \geq 1} j \xi_j(t) = K \exp\left\{(\lambda\theta - \mu - \kappa)t - \lambda\theta \int_0^t \sum_{j \geq 1} \xi_j(u) du\right\}.$$

Remarks on Theorems 4.3 and 4.25 The following remarks are true for model DN (choose $\kappa = 0$ in model DNM) and model DNM. Where another theorem is quoted or changes are nevertheless necessary, the remarks for Theorem 4.3 follow in brackets [.]

1) Comparing with (4.15) [(4.7) for Theorem 4.3], we see that the above equation has a linear part $K \exp\{(\lambda\theta - \mu - \kappa)t\}$ and a non-linear correction term of the form $\exp\{-\lambda\theta \int_0^t \sum_{j \geq 1} \xi_j(u) du\}$. If we start with a small proportion of infected and the parameters are such that the epidemic develops, this non-linear correction becomes increasingly important. Further, as $\sum_{j \geq 1} \xi_j = 1 - \xi_0$, if the proportion of uninfected ξ_0 is smaller than $R_0^{-1} = (\mu + \kappa)/(\lambda\theta)$, then the average number of parasites per individual declines and if $\xi_0 > R_0^{-1}$ the average number of parasites per individual grows. The average number of parasites per individual in the system depends only on the initial average number of parasites per individual, the parameters $(\lambda, \theta, \mu, \kappa)$ and the development of the proportion of uninfected.

2) As $\sum_{j \geq 1} \xi_j(u) \in [0, 1]$, we have

$$\sum_{j \geq 1} j \xi_j(t) \in K[e^{-(\mu + \kappa)t}, e^{(\lambda\theta - \mu - \kappa)t}].$$

3) In a stationary solution with finite number of parasites the average number of parasites per individual is constant. This is included in this formula because in the stationary solution we must have $\bar{\xi}_0 = (\mu + \kappa)/(\lambda\theta)$ according to Theorem 4.26 c) [Theorem 4.4 c)] and this makes the exponential rate 0 as must be.

4) If the proportion of infected were 1, the initial decline of the average number of parasites per individual would be at an exponential rate of $\mu + \kappa$ as can be seen in the above formula too and as is expected to be.

5) We have the same paradoxical behaviour as in the linear models; that is: it is possible that the average number of parasites per individual tends to infinity but that the infection dies out nevertheless. This happens as an

example if $\log \theta \in ((1 + \kappa/\mu)^{-1}, \mu/\kappa)$ [$\theta > e$], $R_0 > 1$ but $R_1 < 1$. As can be seen in Theorem 4.24 Case (2) [Theorem 4.2 Case (2)] the infection dies out, but from Theorem 4.25 [Theorem 4.3] we see that the average number of parasites per individual tends to infinity with such a combination of parameters. In the third region for θ the situation is just the same.

6) It is not possible with any combination of parameters that the proportion of infected converges to 1 because, from Theorem 4.25 [Theorem 4.3], the average number of parasites per individual would converge to 0, a contradiction.

Proof of Theorem 4.25 The theorem follows immediately looking at equation (4.20) in the proof of Theorem 4.23 and taking into account that $\xi_0(u) = 1 - \sum_{j \geq 1} \xi_j(u)$. □

We still have to prove Theorem 4.3.

Proof of Theorem 4.3 The proof of Theorem 4.3 is the proof of Theorem 4.25 if $\mu - \kappa$ is everywhere replaced by μ . The proof of Theorem 4.23 stays valid if $\kappa = 0$ as can be seen in Barbour and Kafetzaki (1993), proof of Theorem 4.1. □

The next theorem summarises all results about stationary solutions in model DNM:

Theorem 4.26 *a) In every non-linear system DNM we always have the trivial stationary solution $\bar{\xi} = e_0$ no matter which values the parameters take.*

b) There is no nontrivial stationary solution of DNM with finite average number of parasites per individual if $\log \theta \geq (1 + \kappa/\mu)^{-1}$.

c) Suppose that $\log \theta < (1 + \kappa/\mu)^{-1}$ and $R_0 > 1$. Then there exists a unique stationary solution $\bar{\xi}$ of DNM with finite average number of parasites per individual. For this stationary solution we furthermore have $\bar{\xi}_0 = R_0^{-1}$.

d) Assuming the conditions of c) and as long as R_0 remains greater than 1, the ratios $\bar{\xi}_j/(1 - \bar{\xi}_0)$ for $j \geq 1$ do not change if the vector (λ, μ, κ) is altered in such a way that the ratio κ/μ remains constant. More, if $p_{10} + p_{11} < 1$ then these ratios can not all stay the same if the ratio κ/μ is altered.

Remark 1. We do not know anything about stationary solutions with infinite average number of parasites per individual (see “4.6 Open questions” 4.6.3.).

2. We do not know whether in the situation of Theorem 4.26 c) a solution ξ does converge towards $\bar{\xi}$ (see “4.6 Open questions” 4.6.2.).

3. Result d) is consistent with result d) of Theorem 4.4 if we choose $\kappa = 0$ because then the ratio κ/μ equals 0 in any case.

Proof of Theorem 4.26 a) is obvious.

b) In this part we assume that $\log \theta \geq 1/(1 + \kappa/\mu)$. We prove part b) by contradiction: we show that if we have a nontrivial stationary solution $\bar{\xi}$ of DNM with finite average number of parasites per individual, then we must have a nontrivial stationary solution $\bar{\Xi}$ of DLM which is contradictory to Theorem 4.20 b). So let us suppose that $\bar{\xi}$ is a nontrivial stationary solution of DNM with finite average number of parasites per individual. Therefore, if we put $\bar{\xi}$ in the right side of DNM we get zero. We once again repeat the equations DNM which are as follows:

$$\begin{aligned} \frac{d\xi_j}{dt} &= (j+1)\mu\xi_{j+1} - j\mu\xi_j + \lambda\xi_0 \sum_{l \geq 1} x_l p_{lj} - \kappa\xi_j; \quad j \geq 1, \\ \frac{d\xi_0}{dt} &= \mu\xi_1 - \lambda\xi_0(1 - \sum_{l \geq 0} \xi_l p_{l0}) + \kappa(1 - \xi_0). \end{aligned} \quad (\text{DNM})$$

Now we write the equations DLM (with a λ') and compare these two systems with each other.

$$\frac{d\Xi_j}{dt} = (j+1)\mu\Xi_{j+1} - j\mu\Xi_j + \lambda' \sum_{l \geq 1} \Xi_l p_{lj} - \kappa\Xi_j; \quad j \geq 1. \quad (\text{DLM})$$

As we are only interested in a stationary solution, we have a constant $\bar{\xi}_0$ in DNM in the infection process. But then, if we choose $\lambda' := \lambda\bar{\xi}_0$, the equations are the same in DNM and DLM for $j \geq 1$. So if we have a stationary solution $\bar{\xi}$ of DNM with finite average number of parasites per individual, then with the choice $\bar{\Xi}_j := \bar{\xi}_j$ for $j \geq 1$ we have a stationary solution for DLM with finite average number of parasites per individual. This is contradictory to Theorem 4.20 b).

c) Let us first construct a candidate g for the unique nontrivial stationary solution of DNM with finite average number of parasites per individual in the following way: We choose $g_0 := R_0^{-1}$ (< 1). Then we define $\lambda' := \lambda g_0$. We now have

$$\frac{\lambda'\theta}{\mu + \kappa} = \frac{\lambda g_0 \theta}{\mu + \kappa} = 1$$

as $g_0 = R_0^{-1}$. We know by Theorem 4.20 c) that if $\log \theta < 1/(1 + \kappa/\mu)$, there exists a nontrivial stationary solution $\bar{\Xi}$ of DLM which is unique up to scalar multiplication. We choose that unique nontrivial stationary solution $\bar{\Xi}^*$ of DLM which is scaled such that

$$g_0 + \sum_{j \geq 1} \bar{\Xi}_j^* = 1.$$

Our candidate is the g such that $g_0 = R_0^{-1}$ as chosen above and then we choose $g_j := \bar{\Xi}_j^*$ for $j \geq 1$.

We now have to check that this candidate satisfies our demands: From Theorem 4.20 c) g inherits nontriviality and that the number of parasites is finite. Additionally we have chosen $g_0 = R_0^{-1}$ which solves one part of c). We therefore only have to prove that g is a stationary solution of DNM and that it is unique under the constraints above. Let us look at the two systems DNM and DLM (repeated in the proof of part b)). $g_j, j \geq 1$ is a stationary solution of DLM. As λ' is by construction equal to λg_0 and g_0 is constant, the two systems are even equivalent for $j \geq 1$. So g does satisfy all equations of DNM for $j \geq 1$ too. We have to check the $j = 0$ -equation too. But as the right side of DNM sums up to 0 this equation must be satisfied too. Therefore we have a stationary solution. We now have to show that it is unique amongst the nontrivial stationary solutions with finite average number of parasites per individual. We prove this through contradiction. Suppose we have two different nontrivial stationary solutions p and q of system DNM with finite average number of parasites per individual. We now construct two different nontrivial stationary solutions p' and q' of a system DLM with parameters $(\tilde{\lambda}, \theta, \mu, \kappa)$ where p' is not a scalar multiple of q' . But this is contradictory to Theorem 4.20 c). In fact we can simply choose $p'_j := p_j$ for $j \geq 1$ and $q'_j := q_j$ for $j \geq 1$. We choose $\tilde{\lambda} := \lambda p_0$. p' is a stationary solution of system DLM with parameters $(\tilde{\lambda}, \theta, \mu, \kappa)$ because of the equivalence of systems DNM and DLM if we choose the ξ_0 in the infection process of DNM to be constant (as it is in a stationary solution). The same construction can be carried out with q' . We choose $\bar{\lambda} := \lambda q_0$. q' is a stationary solution of system DLM with parameters $(\bar{\lambda}, \theta, \mu, \kappa)$ because of the equivalence of systems DNM and DLM if we choose the ξ_0 in the infection process of DNM to be constant (as it is in a stationary solution). In system DLM we can only have a nontrivial stationary solution if $R_0 = 1$ because of equation (4.15). But this must be true for both combinations of parameters:

$$R_0 = \frac{\tilde{\lambda}\theta}{\mu + \kappa} = \frac{\bar{\lambda}\theta}{\mu + \kappa} = 1,$$

and therefore we must have $\tilde{\lambda} = \bar{\lambda}$ and $p_0 = q_0$. So in fact we have two different stationary solutions p' and q' of the same system DLM with parameters $(\tilde{\lambda}, \theta, \mu, \kappa)$. They both sum up to $(1 - p_0)$ which shows that neither is a scalar multiple of the other. So we have two nontrivial stationary solutions of DLM with finite numbers of parasites where neither is a scalar multiple of the other, a contradiction to Theorem 4.20 c).

d) Let u be the unique stationary solution of DNM with parameters $(\lambda, \theta, \mu, \kappa)$ and let v be the unique stationary solution of DNM with altered parameters $(\alpha\lambda, \theta, \beta\mu, \gamma\kappa)$ where α, β and γ are each positive. We define $R_0 := \lambda\theta/(\mu + \kappa)$ and $R_* := \alpha\lambda\theta/(\beta\mu + \gamma\kappa)$. We want to show that $u_j/(1 - u_0) = v_j/(1 - v_0)$ for all $j \geq 1$ under the assumptions of Theorem 4.26 d) (if $\beta = \gamma$). We show this by proving that the proportions amongst the $u_j, j \geq 1$, are the

same as the proportions amongst the $v_j, j \geq 1$. In a stationary solution the derivatives are all 0. So u must satisfy

$$0 = (j+1)\mu u_{j+1} - j\mu u_j + \frac{\mu + \kappa}{\theta} \sum_{l \geq 1} u_l p_{lj} - \kappa u_j, \quad (4.22)$$

for all $j \geq 1$, (we used $u_0 = R_0^{-1} = (\mu + \kappa)/(\lambda\theta)$ from part c)) and v must satisfy

$$0 = (j+1)\beta\mu v_{j+1} - j\beta\mu v_j + \frac{\beta\mu + \gamma\kappa}{\theta} \sum_{l \geq 1} v_l p_{lj} - \gamma\kappa v_j, \quad (4.23)$$

for all $j \geq 1$ (we used $v_0 = R_*^{-1} = (\beta\mu + \gamma\kappa)/(\alpha\lambda\theta)$ from part c) again). To assume that the ratio κ/μ remains constant means that $\beta = \gamma$. So in fact, equation (4.23) is equation (4.22) multiplied by $\beta \neq 0$. But as we are looking at stationary solutions, and so the u_0 and v_0 respectively are constant, these equations are in fact both linear and so multiplying with a constant does not change their solutions. Therefore u satisfies equation (4.23) too. But stationary solutions of a system of type DLM are unique up to scalar multiplication by Theorem 4.20 c). So both solutions must be equal up to scalar multiplication, hence the proportions amongst their co-ordinates must be the same too. So the first part of Theorem 4.26 d) is proved: the ratios $\bar{\xi}_j/(1 - \bar{\xi}_0)$ for $j \geq 1$ do not change if the vector (λ, μ, κ) is altered in such a way that the ratio κ/μ remains constant and R_0 remains larger than 1.

We are now going to rule out the possibility of other changes. So suppose that the equations $u_j = v_j(1 - u_0)/(1 - v_0)$ hold for all $j \geq 1$, that means that the ratios $u_j/(1 - u_0)$ do not change if the parameters are altered. We must now show that this implies $\beta = \gamma$. But the assumption $u_j = v_j(1 - u_0)/(1 - v_0)$ for all $j \geq 1$ means that u is a scalar multiple of v . So u must satisfy equations (4.23) too. We can write equations (4.22) and (4.23) in a more convenient form:

$$u_j = \frac{(j+1)\mu u_{j+1} + \frac{\mu + \kappa}{\theta} \sum_{l \geq 1} u_l p_{lj}}{j\mu + \kappa}, \quad (4.24)$$

for all $j \geq 1$. In the same way v (and as we have just seen u too therefore) must satisfy the following equation:

$$v_j = \frac{(j+1)\beta\mu v_{j+1} + \frac{\beta\mu + \gamma\kappa}{\theta} \sum_{l \geq 1} v_l p_{lj}}{j\beta\mu + \gamma\kappa}, \quad (4.25)$$

for all $j \geq 1$. Let us write A_j for $\sum_{l \geq 1} u_l p_{lj}$. As u satisfies (4.24) and (4.25) we have

$$\frac{(j+1)\mu u_{j+1} + \frac{\mu + \kappa}{\theta} A_j}{j\mu + \kappa} = \frac{(j+1)\beta\mu u_{j+1} + \frac{\beta\mu + \gamma\kappa}{\theta} A_j}{j\beta\mu + \gamma\kappa}$$

for all $j \geq 1$. Simple calculations lead to

$$\beta[\kappa\theta(j+1)\mu u_{j+1} + A_j(\mu\kappa - j\mu\kappa)] = \gamma[\kappa\theta(j+1)\mu u_{j+1} + A_j(\mu\kappa - j\mu\kappa)]$$

for all $j \geq 1$.

But $\kappa\theta(j+1)\mu u_{j+1} + A_j(\mu\kappa - j\mu\kappa)$ is not 0 for all $j \geq 1$ as can be seen in the equation for $j = 1$ because $u_2 > 0$ in a stationary solution of DLM if $p_{10} + p_{11} < 1$. So $\beta = \gamma$ must hold which finishes the proof. \square

The only result we found about convergence towards a nontrivial stationary solution of DNM is

Theorem 4.27 *Let $\log \theta \geq (1 + \kappa/\mu)^{-1}$ and $\bar{\xi}$ be a nontrivial stationary solution of DNM. For a solution ξ of DNM with initial conditions $\xi(0) = y$ where $\sum_{j \geq 1} jy_j < \infty$ we have the following behaviour:*

Case 1) $(1 + \kappa/\mu)^{-1} \leq \log \theta \leq \mu/\kappa$: If $R_1 > 1$, then $\lim_{t \rightarrow \infty} \xi(t) = \bar{\xi}$ is only possible if $\bar{\xi}_0 \geq 1/R_1$.

Case 2) $\mu/\kappa < \log \theta$: If $R_2 > 1$, then $\lim_{t \rightarrow \infty} \xi(t) = \bar{\xi}$ is only possible if $\bar{\xi}_0 \geq 1/R_2$.

Proof of Theorem 4.27 We use inequality (4.21) of the proof of Theorem 4.24, that is

$$m_\alpha^\infty(t) \leq m_\alpha^\infty(v) - \int_v^t (\mu\alpha + \kappa - \lambda\theta^\alpha \xi_0) m_\alpha^\infty(u) du, \quad (4.26)$$

for $0 \leq v \leq t$ where $m_\alpha^\infty(t) = \sum_{j \geq 1} j^\alpha \xi_j(t)$. Let us assume that $R_1 > 1$ (or $R_2 > 1$ respectively in the third region for θ) and $\lim_{t \rightarrow \infty} \xi(t) = \bar{\xi}$. If there exists an $\alpha \in [0, 1]$ such that $g(\alpha) := (\mu\alpha + \kappa - \lambda\theta^\alpha \bar{\xi}_0) > 0$, we can deduce by (4.26) and the Gronwall-inequality that $m_\alpha^\infty(t)$ converges towards 0 for $t \rightarrow \infty$. Then ξ must converge towards the trivial solution e_0 too which is a contradiction to our assumptions. In both cases we prove the existence of such an α by using $\bar{\xi}_0 < 1/R_1 < 1$ (or $\bar{\xi}_0 < 1/R_2 < 1$ respectively in the third region for θ). With these contradictions we then have finished the proofs.

Let us first assume that $1/(1 + \kappa/\mu) \leq \log \theta \leq \mu/\kappa$ and $\bar{\xi}_0 < 1/R_1 < 1$. We choose

$$\alpha_1 := \min\left(1, \frac{1}{\log \theta} \log\left(\frac{\mu}{\lambda \bar{\xi}_0 \log \theta}\right)\right).$$

Let us first treat the case where $\alpha_1 = (1/\log \theta) \log(\mu/(\lambda \bar{\xi}_0 \log \theta))$. Then α_1 must be smaller or equal to 1 and because $1/(1 + \kappa/\mu) \leq \log \theta \leq \mu/\kappa$ we have $\alpha_1 \geq 0$. We now must check that $g(\alpha_1) > 0$. We have

$$g(\alpha_1) = \frac{\mu}{\log \theta} \log\left(\frac{\mu}{\lambda \bar{\xi}_0 \log \theta}\right) + \kappa - \frac{\mu}{\log \theta}.$$

This is larger than 0 if

$$\log\left(\frac{\mu}{\lambda \bar{\xi}_0 \log \theta}\right) + \log\left(\theta^{\frac{\kappa}{\mu}}\right) > 1,$$

which is satisfied if

$$\frac{\mu\theta^{\kappa/\mu}}{\lambda\bar{\xi}_0 \log \theta} > e.$$

As $R_1^{-1} > \bar{\xi}_0$ this inequality is satisfied. Now we treat the case where $\alpha_1 = 1$ and therefore we may additionally use that $(1/\log \theta) \log(\mu/(\lambda\bar{\xi}_0 \log \theta)) > 1$. This is equivalent to $\mu/(\lambda\theta \log \theta) > \bar{\xi}_0$. Therefore $\bar{\xi}_0 < \min(\mu/(\lambda\theta \log \theta), R_1^{-1})$. Because $\log \theta \geq 1/(1 + \kappa/\mu)$, we have $\mu/(\lambda\theta \log \theta) < R_1^{-1}$. Hence

$$\bar{\xi}_0 < \frac{\mu}{\lambda\theta \log \theta}. \quad (4.27)$$

Let us now show that $g(1) = \mu + \kappa - \lambda\theta\bar{\xi}_0 > 0$. By (4.27) this is satisfied if $\mu + \kappa \geq \mu/\log \theta$. But this is satisfied because $\log \theta \geq 1/(1 + \kappa/\mu)$ which ends the proof of the first case.

Now we assume that $\log \theta > \mu/\kappa$ and $\bar{\xi}_0 < 1/R_2 < 1$. Define $c := 1/R_2 - \bar{\xi}_0 > 0$. Then we have

$$g(\alpha) = \mu\alpha + \kappa - \lambda\theta^\alpha\bar{\xi}_0 = \mu\alpha + \kappa - \lambda\theta^\alpha(\kappa/\lambda - c) = \mu\alpha + \kappa - \theta^\alpha\kappa + \lambda\theta^\alpha c,$$

where we used the definitions of c and $R_2 = \lambda/\mu$. This shows that there exists an $\alpha \in [0, 1]$ such that $g(\alpha) > 0$ (let α tend to 0). This ends the proof of case 2). □

Remarks on the basic reproduction ratios V As the results of the models without mortality and with mortality of individuals are analogous, we present them together in this summary. We present the results for the models including mortality and the analogous results for the models without mortality are added in brackets [.] where necessary.

To begin with we want to remember that in the non-linear stochastic model with mortality of humans the epidemic finally dies out with probability one no matter what values the parameters have (Theorem 2.6 [Theorem 2.2]).

In the deterministic models with nontrivial initial values there are exactly two possibilities for the development of the disease: either the epidemic dies out or it develops. In case the epidemic dies out we must make clear that *in finite time* the number (or proportion) of infected individuals never equals exactly 0. But in the limit (as time goes to infinity) this is so.

We have to add one remark about the border of the first to the second region of θ : If $\log \theta = 1/(1 + \kappa/\mu)$, then we have $R_0 = R_1$ [if $\theta = e$ then $R_0^{(0)} = R_1^{(0)}$]. So for example in Theorems 4.24 [Theorem 4.2] it does not matter whether we include the case $\log \theta = 1/(1 + \kappa/\mu)$ in case 1) or 2). But due to the behaviour of the auxiliary Markov process Y described in Theorem 4.10,

comparing for example Theorem 4.26 b) and c), we see that for $\log \theta = (1 + \kappa/\mu)$ we have no nontrivial stationary solution of DNM with finite average number of parasites per individual. Additionally we have $R_1 = R_2$ if $\log \theta = \mu/\kappa$. So again, in Theorem 4.24 it does not matter whether we include the case $\log \theta = \mu/\kappa$ in case 2) or 3).

In previous parts of chapter 2 which we titled with “Remarks on the basic reproduction ratios I, II, III, and IV”, we partly explained the meaning of the relevant combinations R_i of parameters. We know the meaning of R_0 : This combination of parameters stands for the average number of offspring of a parasite during his whole lifetime in the absence of density constraints. “Absence of density constraints” means that each contact of an infected person is made with an uninfected person; mathematically: the contact rate λ is not decreased to the effective contact rate $\lambda\xi_0(t)$. In the linear models we do not have these density constraints. But in the non-linear models λ is decreased to the effective contact rate. Suppose we are looking at a non-linear model and $R_0 > 1$ but $R_0\xi_0(t_1) < 1$. So the average number of parasites per individual declines (see Remark 1 to Theorem 4.25 [Remark 1 to Theorem 4.3]) and we could expect that the epidemic dies out. But at some time t_2 when the epidemic is near to extinction, $\xi_0(t_2)$ becomes almost one and so the effective contact rate gets near to the unaltered contact rate λ and then $R_0\xi_0(t_2) > 1$ again. Then the epidemic develops again (at least the average number of parasites per individual grows again!). Intuitively it is therefore clear, that $R_0 = 1$ must be the threshold condition for the development of the average number of parasites per individual in the entire system. Therefore R_0 would be called “Basic Reproduction Ratio” in the common theory of epidemiology. But in our models it is not so that the development of the epidemic can be described simply with R_0 in a satisfactory manner.

Summarised, the results are as follows: In the non-linear model DNM *and* in the linear model DLM the **threshold results** are the same mutatis mutandis (Theorem 4.24 and Remark 4 to Theorem 4.18 [Theorem 4.2 and Remark 1 to Theorem 4.8]): If $\log \theta \leq 1/(1 + \kappa/\mu)$, the proportion (or number in DL) of individuals tends to 0 if $R_0 < 1$ and if $R_0 > 1$ the proportion (or number) of individuals does not tend to 0. If $1/(1 + \kappa/\mu) < \log \theta \leq \mu/\kappa$, the proportion (or number) of individuals tends to 0 if $R_1 < 1$ and if $R_1 > 1$ the proportion (or number) of individuals does not tend to 0. If $\log \theta > \mu/\kappa$, the proportion (or number) of individuals tends to 0 if $R_2 < 1$ and if $R_2 > 1$ the proportion (or number) of individuals does not tend to 0. In comparison to the models without mortality of humans a third region was added. Naturally, there is a difference between the two models DNM and DLM concerning the **exact number of parasites**: According to Theorem 4.23 [Theorem 4.1] the average number of parasites per individual tends to 0 in DNM if $R_0 < 1$ for all θ . The average number of parasites per individual develops according to Theorem 4.25

[Theorem 4.3] in model DNM and is therefore mainly governed by R_0 being larger or smaller than 1. It is especially possible that the average number of parasites per individual tends to infinity but the epidemic dies out nevertheless (Remark 5 to Theorem 4.25 [Remark 5 to Theorem 4.3]). The analogous result to Theorem 4.25 [Theorem 4.3] in DLM is equation (4.15) [equation (4.7)]. In the linear case it is R_0 that governs the development of the number of parasites exclusively. For **stationary solutions** it is necessary that $R_0 > 1$ in DNM and $R_0 = 1$ in DLM for to have interesting results.

With the exception of Theorem 2.7 [Theorem 2.2], the results in the deterministic environment are analogous to the results of the stochastic approaches (Theorems 2.8 and 2.10 [Theorems 2.3 and 2.5]). But there the number of parasites must be replaced by the *expected* number of parasites in the linear model and we do not have an analogous result for the non-linear model. It is clear that we do not have any nontrivial ($= e_0$ or 0 respectively) stationary solutions or periodic solutions in the stochastic models.

We have seen that in all deterministic models there are combinations of parameters which allow for the numbers of parasites to tend to infinity while the infection dies out nevertheless. We want to make clear that this is not possible in the stochastic models; there (if the parameters are in that critical region) only the *expectation* of the number of parasites converges to infinity. This difference occurs because in the deterministic models there can well be positive real numbers $\sum_{j \geq 1} \xi_j(t)$ tending to zero as $t \rightarrow \infty$ but $\sum_{j \geq 1} j \xi_j(t)$ tends to infinity at the same time. This is not possible in the stochastic environment because we only have whole positive numbers.

4.6 Open questions

1. In view of Theorem 4.2 [Theorem 4.24 in model DNM], do the infections die out in model DN [DNM] if the relevant $R_i^{(0)}$ [R_i] equal exactly 1?

2. In view of Theorem 4.4 c) ($\theta < e$ and $R_0^{(0)} > 1$) [Theorem 4.26 c) ($\log \theta < 1/(1 + \kappa/\mu)$ and $R_0 > 1$) in model DNM], do we have convergence in DN [DNM] towards that stationary solution $\bar{\xi}^{(0)}$ [$\bar{\xi}$]?

3. In all 4 deterministic models: Are there any stationary solutions $\bar{\xi}$ with infinite average number of parasites per individual and under which conditions does a solution ξ converge to $\bar{\xi}$? Is such a solution unique?

4. Are there any stationary solutions $\bar{\Xi}^{(0)}$ [$\bar{\Xi}$ in DLM] of DL [DLM] such that $\sum_{j \geq 1} \bar{\Xi}_j^{(0)} = \infty$ [$\sum_{j \geq 1} \bar{\Xi}_j = \infty$ in DLM] and under which conditions does a solution $\Xi^{(0)}$ [Ξ in DLM] converge to $\bar{\Xi}^{(0)}$ [$\bar{\Xi}$ in DLM]? Is such a solution unique?